Topics in percolation theory

Kiss, D.

2013

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Cover: simulation of a N-parameter frozen percolation configuration at time 1. Blue hexagons correspond to frozen clusters, while the red ones are non-frozen open hexagons.

The research in this thesis has been carried out at the Centrum Wiskunde & Informatica (CWI), the national research institute for mathematics and computer science in the Netherlands. It has been financially supported by the Netherlands Organization for Scientific Research (NWO).

VRIJE UNIVERSITEIT

Topics in percolation theory

ACADEMISCH PROEFSCHRIFT

ter verkrijging van de graad Doctor aan
de Vrije Universiteit Amsterdam,
op gezag van de rector magnificus
prof.dr. F.A. van der Duyn Schouten,
in het openbaar te verdedigen
ten overstaan van de promotiecommissie
van de Faculteit der Exacte Wetenschappen
op woensdag 3 juli 2013 om 13.45 uur
in de aula van de universiteit,
De Boelelaan 1105

door

Demeter Kiss

geboren te Miskolc, Hongarije

promotor: prof.dr. J. van den Berg

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1 Introduction

Since its introduction in 1957, the percolation model received significant attention in the mathematical community. From a mathematician's point of view, the model is simple and it is a good source of interesting problems which are easy to state, but, in most cases, are quite challenging to solve. Moreover, in the study of the model, tools from a large variety of mathematical disciplines are required such as probability theory, combinatorics, analysis and geometry. As in the other statistical physics models, the main question is how local (microscopic) interactions between small particles manifest on large (macroscopic) scale. In particular, the percolation model has a phase transition similar to the phenomenon observed when ice melts to water. This makes the model worthy of studying, despite its simplicity. Moreover, it can be used as a starting point for constructing more delicate models.

Since 1957, substantial progress was made in the study of the model, many related models were introduced and a large variety of tools were developed to study these models, hence the theory of percolation has born. The results established amount to hundreds, if not thousands of research papers. We do not give an overview of these results – we only introduce and discuss the concepts which are relevant for our purposes herein. The reader is referred to the books [66], [50] and [19] for a review.

The aim of the thesis to contribute to the theory of percolation. We investigate some related models in detail and some mathematical tools used in the study of these models. The connection of our results to this theory are not in all cases immediate. We make these clear and motivate our results in this introduction.

We start with a discussion of the percolation model in Section 1.1. One of the tools used in the study of phase transition of the percolation model are the sharp threshold results. In Section 1.2 we explore some of these results and make connections to concentration inequalities similar to the one of Talagrand [90]. In Section 1.3 we introduce the first passage percolation model and its weakly dependent version. In Section 1.4, we turn to the so called frozen percolation model and discuss some of its properties. The arguments in the introduction are non-rigorous since our aim is to give a general impression about the subjects, rather then going into delicate technical details. The notation used here is slightly different from the one used in the later chapters.

Our contributions to the field are in Chapter 2-5. In Chapter 2, we state and prove a generalization of Talagrand's inequality for products of finite probability spaces. In Chapter 3 we apply this generalization for certain 'coding' variables and deduce a sublinear variance bound for the travel-time in weakly dependent first passage percolation models. In Chapter 4, we consider the N-parameter frozen percolation process on the binary tree. We show that it converges in some weak sense to the ∞-parameter process as N tends to infinity. Finally, in Chapter 5, we consider the N-parameter process on the triangular lattice and deduce its properties. In particular, we answer the questions of van den Berg, de Lima and Nolin [94]. Our results lead to a conjecture on the scaling limit of the N-parameter model, which seems to be a promising direction for future research.

The thesis is based on the following papers:

- Chapter 2: Demeter Kiss. A generalization of Talagrand's variance bound in terms of influences. preprint, arXiv:1007.0677, 2010.
- Chapter 3: Jacob van den Berg and Demeter Kiss. Sublinearity of the travel-time variance for dependent fist passage percolation. *Annals of Probability,* 40(2):743-764, 2012.
- Chapter 4: Jacob van den Berg, Demeter Kiss and Pierre Nolin. A percolation process on the binary tree where large finite clusters are frozen. *Electronic Communications in Probability,* 17(2):1-11, 2012.
- Chapter 5: Demeter Kiss. Frozen percolation in two dimensions. preprint, arXiv: 1302.6727, 2012.

1.1 Percolation model

As announced above, we only consider the concepts of the percolation model which are relevant for our purposes. In Section 1.1.1 we introduce the model and examine its key property, the phase transition. This examination leads us to Section 1.1.2, where we indicate how sharp threshold can be used to prove the phase transition. We further investigate these results in Section 1.2. In Section 1.1.3 we describe the critical and near critical states in more detail using scaling limits. The results in Section 1.1.3 will be used in Section 1.4.

1.1.1 Model description and phase transition

We follow the interpretation of the model by Grimmett [50]: Consider a large porous rock (similar to Maasdam cheese, but with much smaller holes) in a bucket of water. Does the water percolate to the middle of the rock? Hammersley and Broadbent [25] modelled the porous medium (the rock) by a regular structure of interconnecting tubes where, independently from each other, each tube is wide enough to let the water pass through (open) with probability p, and it is not (closed) with probability $1-p$. We represent the regular structure by a graph G, where the tubes correspond to the edges of G. We are mainly interested in the case where G is a lattice, in particular where $G = \mathbb{Z}^d$ for some $d \in \mathbb{N}$. For the discussion of this section we restrict to the case where $d = 2$, that is $G = \mathbb{Z}^2$.

For $p \in [0,1]$ let $\mathbb{P}_p$ denote the probability distribution corresponding to the percolation model with parameter p. Let

$$\theta(p) := \mathbb{P}_p\left(O \leftrightarrow \infty\right) \tag{1.1.1}$$

denote the probability that the origin has an open path to infinity. We will also use the notation

$$\bar{\theta}\left(p\right) := \mathbb{P}_p\left(\text{there is an infinite open cluster}\right) \tag{1.1.2}$$

The pores in the rock are small compared to the rock, and we may reformulate the main question in mathematical terms by asking for which values of p is $\theta\left(p\right) > 0$?

Before we answer this question, we start with an observation: There is a natural coupling between the percolation models on the same graph with different parameters, described as follows. For each edge e, independently from each other, we assign a random value τ_e with uniform distribution on $[0,1]$. For $p \in [0,1]$ we declare an edge p-open (p-closed), if $\tau_e \le p$ ($\tau_e > p$). Hence the set of p-open edges have the same distribution as the open edges in the percolation model with parameter p. This coupling shows that $\theta(p)$ as a function of p is non-decreasing. This implies that there is a critical value $p_c \in [0,1]$ such that

- $\theta\left(p\right) = 0$ for $p < p_c$, and
- $\theta\left(p\right) > 0$ for $p > p_c$.

It is a simple exercise, using Kolmogorov's 0-1 law, to show that

$$\bar{\theta}\left(p\right) = \begin{cases} 0 & \text{for } p < p_c \\ 1 & \text{for } p > p_c. \end{cases} \tag{1.1.3}$$

A couple of years after the introduction of the model, Harris [54] showed that for $p \le 1/2$, there is no infinite open cluster with probability 1. Hence $p_c \ge 1/2$. It was only much later when Kesten [66] proved that for $p > 1/2$ there is an infinite cluster almost surely. This combined with the result of Harris shows that $p_c = 1/2$, and answers our question above.

1.1.2 Application of sharp threshold results

Let us further investigate the model. Let $B(n)$ denote the box of with radius $n \in \mathbb{N}$, that is

$$B(n) := \left\{-n, -n+1, \ldots, n-1, n\right\}^2. \tag{1.1.4}$$

Let $a, b, c, d \in \mathbb{Z}$. For a rectangle

$$R = ([a,b] \times [c,d]) \cap \mathbb{Z}^2$$

let $\mathcal{H}_o(R)$ $(\mathcal{H}_c(R))$ denote the event that there is an open (closed) horizontal crossing in R, i.e there is an open path in R connecting the left and the right sides of R.

Remark. With a slight abuse of notation, we define

$$\mathcal{H}_i([a,b]\times[c,d]) := \mathcal{H}_i(R)$$

for $i \in \{o,c\}$.

The event $\mathcal{H}_o(B(n))$ for some large $n \in \mathbb{N}$, roughly speaking, is a finite version of the event $\{\text{there is an infinite open cluster}\}$. Hence we expect that the probability $\mathbb{P}_p(\mathcal{H}_o(B(n)))$ seen as a function of p, has a behaviour similar to $\bar{\theta}(p)$ described in (1.1.3). Note that the quantity $\mathbb{P}_p(\mathcal{H}_o(B(n)))$ is a polynomial in p, hence it cannot have jumps. Nevertheless, we expect that close to $p_c = 1/2$, it increases quite steeply. Moreover, the function $p \to \mathbb{P}_p(\mathcal{H}_o(B(n)))$ close to $1/2$ should become steeper and steeper as we increase n. Russo [85] called this property an approximate 0-1 law in contrast to the 0-1 law (1.1.3). Later, this property became know in the literature as sharp transition or sharp threshold.

By reversing the arguments above, we can give a proof of Kesten's result ($p_c \le 1/2$): Russo [85] (as well as Kesten [66] in some sense) showed that the function $p \to \mathbb{P}_p(\mathcal{H}_o(B(n)))$ has an approximate 0-1 law near $1/2$ as described above. Thus Russo put the proof of $p_c = 1/2$ in a more general framework. This strategy was later rediscovered and refined by Bollobás and Riordan in [18]. They also applied this strategy to Voronoi percolation in [20] and percolation on random Johnson-Mehl tessellations [21]. A key difference between Russo's proof [85] and the proof of Bollobás and Riordan [18] is in the tools they use to derive that $\mathbb{P}_p(\mathcal{H}_o(B(n)))$ has a sharp transition at $1/2$: Roughly speaking, Russo in [85] proved a general theorem. He applied it to $\mathbb{P}_p(\mathcal{H}_o(B(n)))$, and deduced that $\mathbb{P}_p(\mathcal{H}_o(B(n)))$ has a sharp transition at $1/2$ if the influence of each edge in $B(n)$ on the event $\mathcal{H}_o(B(n))$ is small in the percolation model with parameter $p \approx 1/2$. Bollobás and Riordan [18] showed this result by using a theorem of Friedgut and Kalai [42] which is based on [62] and [24]. This result, roughly speaking, states that every symmetric increasing event has a sharp transition. As we can see, these sharp threshold results are quite useful tools in the study of percolation. This leads us to Section 1.2 where we describe them in more detail.

1.1.3 Critical and near critical scaling limits

As we mentioned in the introduction, we are particularly interested in the large scale (macroscopic) behaviour of the percolation model with parameter $p \in [0,1]$. In the lines above, we considered the existence of an infinite cluster. In the following we refine our analysis, and turn to the properties of the large but finite open clusters. We have three cases depending on the value of p. When $p < p_c$, then, roughly speaking, the diameter of the open clusters have an exponential tail. The same holds for the diameter of the *finite* open clusters when $p > p_c$:

Definition 1.1.1. *Let $\mathcal{C}(v)$ denote the open cluster (open connected component) of a vertex v. In the case $v = 0$, we write $\mathcal{C} := \mathcal{C}(0)$.*

The following result follows from Theorem 2 of [66], or alternatively, from the results of Aizenmann and Barsky [3].

Theorem 1.1.2. *Let $p \in [0,1]$, $p \neq p_c = 1/2$. There exists $c = c(p)$ such that*

$$\mathbb{P}_p\left(n \leq diam\left(\mathcal{C}\right) < \infty\right) \leq e^{-cn}.$$

for $n > 0$.

Let $n \in \mathbb{N}$, and consider a percolation configuration with parameter p. Let us scale space by n, that is, decrease the lattice spacing from 1 to $1/n$. When we look at the open clusters intersected with $B(n)$ in this configuration as a set of their vertices, we get subsets of $[-1,1]^2$. Theorem 1.1.2 gives that as $n \to \infty$, the diameter of these sets go to 0 when $p < p_c$. When $p > p_c$ the same holds for the finite open clusters, while the sets corresponding to the infinite cluster converge, in some sense, to the set $[0,1]^2$.

In the following we use the notion of dual graphs. The dual graph of $\mathbb{Z}^2$ denoted by $\mathbb{Z}^{2*}$, is the graph where the vertices of $\mathbb{Z}^{2*}$ are the faces of $\mathbb{Z}^2$, and two vertices of $\mathbb{Z}^{2*}$ are connected if the corresponding faces of $\mathbb{Z}^2$ share an edge. It is easy to check that $\mathbb{Z}^2$ is self-dual, that is, $\mathbb{Z}^2$ and $\mathbb{Z}^{2*}$ are isomorphic. Moreover, there is a one-to-one correspondence between the edges of the dual and primal graph. Hence for a percolation configuration on $\mathbb{Z}^2$ we construct a configuration on $\mathbb{Z}^{2*}$ by assigning the state (open or closed) of the edges of $\mathbb{Z}^2$ to the corresponding dual edges. For $a,b,c,d \in \mathbb{Z}$, let $\mathcal{H}_o^*([a,b] \times [c,d])$ denote the event that there is an open horizontal crossing in the dual graph of $([a,b] \times [c,d]) \cap \mathbb{Z}^2$.

Remark. Note that the dual graph of $([a,b] \times [c,d]) \cap \mathbb{Z}^2$ contains a vertex which corresponds to the unbounded face of $([a,b] \times [c,d]) \cap \mathbb{Z}^2$. We disregard this vertex in the event $\mathcal{H}_o^*([a,b] \times [c,d])$ by forbidding the horizontal crossings to use this vertex.

Let us turn back to the scaling limits of percolation. In the case where $p = p_c$ the situation changes dramatically. Russo [83] and Seymour and Welsh [86] showed the following.

Theorem 1.1.3 (RSW)**.** *Let $a, b > 0$ then there exists $c = c(a,b) > 0$ and $C = C(a,b) < 1$ such that*

$$c < \mathbb{P}_{1/2}\left(\mathcal{H}_o\left([0,an] \times [0,bn]\right)\right) < C \tag{1.1.5}$$

for all $n \in \mathbb{N}$. The analogue of (1.1.5) *holds for closed crossings as well as for dual crossings.*

This, roughly speaking, implies that in the box $B(n)$, there are open and closed clusters with diameter of order n which do not disappear in the scaling limit as n tends to infinity.

At this point, we switch to a closely related model called site percolation on the triangular lattice. That is, instead of the edges, the vertices of the triangular lattice are open or closed. The arguments and the results of Section 1.1 to this point, with straightforward modifications, hold for this model. Moreover, the critical value is $1/2$ for this model as well. The reason to change to this model is technical: the triangular lattice has some additional symmetries compared to the square lattice, which makes certain arguments simpler. See Chapter 5 for more details. These extra symmetries have far reaching consequences: The results in the following are proved only for site percolation on the triangular lattice, and despite the best efforts they are only conjectures for bond percolation on the square lattice.

An open cluster in site percolation on the triangular lattice is surrounded by closed vertices. These closed vertices form some disjoint loops. Instead of the open clusters above, we 'code' the macroscopic features of the model by the collection of these loops. Camia and Newman [28], based on the work of Smirnov, Lawler, Schramm, Werner and others, showed that at $p = p_c$ with the scaling above, the laws of these collections of loops have a limit as $n \to \infty$. There are other ways to describe the macroscopic features and the scaling limit of the percolation model. See [87] and [46] for a review.

In the arguments above, we fixed the parameter $p \in [0, 1]$ and took the limit as the mesh size tends to 0. It is a natural question to ask if there is a scaling limit when the percolation parameter $p = p_n$ depends on the space scale n. In view of the arguments above and Theorem 1.1.2, it turns out that depending on the speed at which $p_n \to p_c$, there are three fundamentally different scaling limits. More precisely, there is a function $r : \mathbb{N} \to \mathbb{N}$ such that depending on the value of

$$\lambda := \lim_{n\to\infty} \frac{p_n - p_c}{r(n)} \tag{1.1.6}$$

we get different scaling limits:

- if $\lambda = 0$, then we get the scaling limit described by Camia and Newman [28],
- if $\lambda = \infty$, ($\lambda = -\infty$,) the we get, in some sense trivial, scaling limit described below Theorem 1.1.2 for $p > p_c$ ($p < p_c$),
- if $\lambda \in \mathbb{R} \setminus \{0\}$ we get the third type, so called near critical scaling limit.

See [27] for a precise description of these scaling limits and the proof of the result above. See [78] and [8] for a proof that the scaling limits above are different (and even singular with respect to each other) for different values of λ.

We rewrite (1.1.6) and define the function

$$p_\lambda(n) := p_c + \lambda r(n) \tag{1.1.7}$$

for $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}$. We derive the precise formula for the function $r(n)$ in two alternative ways in Chapter 5, where (1.1.7) is used explicitly.

We finish this section by mentioning a scaling limit of certain couplings of percolation models. For each fixed n, using the τ values of Section 1.1.2 we couple the percolation models with parameter $p_\lambda(n)$ for $\lambda \in \mathbb{R}$: we get a stochastic process in λ the so called near-critical ensemble. Garban, Pete and Schramm [45, 46] announced a result which states that as $n \to \infty$ these near-critical ensembles, after scaling space by n, converge to a coupling of the near-critical scaling limits described above. This will be interesting for us when we formulate a conjecture on the scaling limit of frozen percolation n Chapter 5.

1.2 From sharp threshold results to hypercontractivity

In Section 1.1.2 we gave some examples of the use of sharp threshold results in percolation theory. As we will see below, these sharp threshold results follow from some more general concentration inequalities. The set-up for these inequalities is broader than the percolation model: We do not need an underlying graph, only an event or a function which depends on some independent random variables. This makes the sharp threshold and concentration results useful in a large variety of disciplines ranging from economics (voting models) to computer science (randomized algorithms).

Our contribution to the field is Chapter 2, where we give generalization of an inequality due to Talagrand to finite probability spaces by extending his original methods in [90]. The proofs in that chapter are almost self-contained, and closely follow the strategy of proof of Talagrand in [90]. We use this result explicitly in Chapter 3. Strictly speaking, our generalization is not a novel result: its formulation is new and we give a new proof for the result. However, as we show below, it follows from the combination of the results of Wolff [99] and Talagrand [90]. Approximately a year after the first appearance of our results, Cordero-Erausquin and Ledoux [30], using different methods, further extended Talagrand's inequality as we explain below.

We organize the section as follows. In Section 1.2.1 we give a brief description of the aforementioned sharp threshold results and their connection to concentration inequalities. In Section 1.2.2 we describe the two key tools used in the proof of the concentration inequalities. The adaptation of these tools are used in the of Talagrand's inequality, which we describe in Section 1.2.3. We informally state our generalization of Talagrand's inequality in Section 1.2.4. For the precise formulation and the proof of our results see Chapter 2. We show the connection between Talagrand's inequality, hypercontractivity and logarithmic Sobolev inequalities. This leads us to the generalization by Cordero-Erausquin and Ledoux [30] which we briefly describe in Section 1.2.5 and finish Section 1.2.

1.2.1 Sharp threshold results and concentration inequalities

Let $n \in \mathbb{N}$. For $p \in [0,1]$ let μ_p denote a measure on $\{0,1\}$ such that $\mu_p(\{1\}) = 1-\mu_p(\{0\}) = p$. Let μ_p^n denote the n-fold product of the measure μ_p. To simplify the notation, we omit the superscript n from μ_p^n in the following.

Definition 1.2.1. *Let $n \in \mathbb{N}$, $A \subset \{0,1\}^n$ and $i \in \{1,2,\ldots,n\}$. Let*

$$\delta_i A := \left\{\omega \in \{0,1\} \middle| \exists \omega' \in A \text{ and } \omega'' \in \{0,1\} \setminus A \text{ such that } \omega_j = \omega'_j = \omega''_j \ \forall j \neq i\right\}.$$

We define the influence of the coordinate i on the event A as

$$I_A(i,p) := \mu_p(\delta_i A).$$

Remark 1.2.2. In percolation theory the coordinates of the vector $\omega \in \{0,1\}^n$ are indexed by the set of edges of the underlying graph. There, for an edge e, the event $\delta_e A$ is usually referred to as 'e is pivotal to A'.

Definition 1.2.3. *Let $N \in \mathbb{N}$. We say that an event $A \subset \{0,1\}^n$ is increasing if for all $\omega \in A$ and $\omega' \in \Omega$ with $\omega_i \leq \omega'_i$ for $i = 1,2,\ldots,n$ we have $\omega' \in A$.*

We start with the approximate 0-1 law of Russo:

Theorem 1.2.4 (Theorem 1 of [85])**.** *For every $\varepsilon > 0$, there exists $\eta > 0$ such that if $A \subset \{0,1\}^n$ is an increasing event, and*

$$\forall i, \forall p \in [0,1], \ I_A(i,p) < \eta,$$

then there exists $p_0 \in [0,1]$ such that

$$\begin{aligned} \mu_p(A) &\leq \varepsilon && \text{for } p \leq p_0 - \eta, \\ \mu_p(A) &\geq 1-\varepsilon && \text{for } p \geq p_0 + \eta. \end{aligned}$$

The statement of Theorem 1.2.4 might be mysterious for the reader for the first sight: one could ask how the probability of the event A is connected to the influences $I_A(i,p)$. The following formula can shed some light on this question. It appeared as Lemma 3 in [84]. See also [74].

Lemma 1.2.5 (Margulis-Russo formula)**.** *Let $A \subseteq \{0,1\}^n$ be an increasing event. Then*

$$\frac{d}{dp}\mu_p(A) = \sum_i I_A(i,p).$$

The formula above does not completely explain Theorem 1.2.4. Note that it follows from a result, which roughly speaking, states that if all the influences $I_A(i,p)$ are small, then their sum is large. Friedgut and Kalai pointed out in [42] that the results in [24] imply the following, so called concentration, result:

Theorem 1.2.6 (Theorem 3.4 of [42]). *There is $c > 0$ such that for all $n \in \mathbb{N}$, $p \in [0,1]$ and $A \subseteq \{0,1\}^n$ we have*

$$\sum_i I_A(i,p) \geq c\mu_p(A)(1-\mu_p(A))\log(1/\eta)$$

when $I_A(i,p) \leq \eta$ for every i.

Remark 1.2.7. Note that in the result above, we did not require that the event A is increasing. Moreover, Theorem 1.2.6 holds for case where the underlying probability space is a product of possibly different probability spaces. That is, roughly speaking, Theorem 1.2.6 holds in the case where the event A depends on some independent, but possibly not identically distributed random variables.

Theorem 1.2.6 implies the following sharp threshold result, which is a quantitative version of Theorem 1.2.4:

Corollary 1.2.8 (Corollary 3.5 of [42]). *Let $A \subseteq \{0,1\}^n$ be an increasing event such that*

$$\forall i, \forall p \in [0,1], \, I_A(i,p) < \eta.$$

If $\mu_p(A) > \varepsilon$, then $\mu_q(A) > 1-\varepsilon$ for $q = p + c\log(1/\varepsilon)/\log(1/\eta)$, where the constant c is universal.

1.2.2 Walsh basis and the Bonami-Beckner inequality

Following Kahn, Kalai and Linial [62], we present the main ideas in the proof of Theorem 1.2.6 in the special case where $p = 1/2$. The key ingredients of this proof are the Walsh basis and the Bonami-Beckner inequality. Let $L = L_{\mu_{1/2}}(\{0,1\}^n)$ denote the (Hilbert) space of real valued functions on $\{0,1\}^n$ with inner product

$$\langle f,g\rangle_{\mu_{1/2}} := \int_{\{0,1\}^n} fg d\mu_{1/2}.$$

For $q \in (0,\infty)$ let

$$\|f\|_q := \left(\int_{\{0,1\}^n} |f|^q \, d\mu_{1/2}\right)^{1/q}.$$

The first tool is the following orthonormal basis of L, called the Walsh basis. It is defined as

$$w_y(x) := \prod_{i=1}^n (-1)^{x_i y_i}, \tag{1.2.1}$$

for $x,y \in \{0,1\}^n$. Let $\mathbf{1}_A$ denote the indicator function of the event A. Let a_y, $y \in \{0,1\}^n$ denote the Fourier coefficients of $\mathbf{1}_A$ in this basis:

$$\mathbf{1}_A = \sum_{y\in\{0,1\}^n} a_y w_y.$$

Then simple computations and Parseval's formula gives that

$$\begin{aligned}\mu_{1/2}(A) &= a_{\underline{0}} \\ &= \sum_{y\in\{0,1\}^n} a_y^2.\end{aligned} \tag{1.2.2}$$

The definition of the Walsh basis (1.2.1) implies that

$$\mathbf{1}_{\delta_i A} = \sum_{y\in\{0,1\}^n : y_i=1} a_y w_y. \tag{1.2.3}$$

This combined with Parseval's formula gives

$$\sum_i I_A(i,p) = \sum_{y\in\{0,1\}^n} [y]\, a_y^2 \tag{1.2.4}$$

where $[x]$ denotes the number of non-zero coordinates of x. (1.2.2) and (1.2.4) shows a connection between the sum of influences and the measure of A. To deduce Theorem 1.2.6, we have to bound the L^2 norm of the operator $S : L \to L$ defined as

$$Sf = \sum_{y\in\{0,1\}^n} [y]\, \hat{f}(y)\, w_y$$

for

$$f = \sum_{y\in\{0,1\}^n} \hat{f}(y)\, w_y. \tag{1.2.5}$$

This can be achieved by the following hypercontractive inequality due to Beckner [10] and Bonami [23]:

Theorem 1.2.9 (Bonami-Beckner inequality)**.** *Let $f \in L$ as in* (1.2.5)*. We define the operator*

$$T_\varepsilon(f) = T_{\varepsilon,n}(f) := \sum_{y\in\{0,1\}^n} \varepsilon^{[y]} \hat{f}(y)\, w_y. \tag{1.2.6}$$

Then

$$\|T_\varepsilon f\|_2 \leq \|f\|_{1+\varepsilon^2}.$$

Combining (1.2.2), (1.2.4) and Theorem 1.2.9 after some lines of computation finishes the proof of Theorem 1.2.6 in the case where $p = 1/2$.

Friedgut and Kalai [42] pointed out that the general case can be obtained from this special case by using two additional ingredients from [24]: approximating random variables with coin-flips, and dominating events with certain increasing events. We do not describe these ingredients here, otherwise we would deviate too much from the main line of our arguments. See [24], [42] and [49] for more details.

The operator (1.2.6) above might seem a bit unnatural at the first sight. However, it has a simple interpretation in terms of random walks which we postpone to Section 1.2.5.

1.2.3 Talagrand's inequality

To this point, we considered bounds on the influences of events. Talagrand in [90] considered a slightly different approach to prove sharp threshold results similar to Theorem 1.2.6. He extended the notion of influences from event to functions, and proved inequalities concerning them. Let $p \in [0,1]$. For a function $f \in L_{\mu_p}\left(\{0,1\}^n\right)$ we define

$$\Delta_i f\left(x_1,\ldots,x_n\right) = f\left(x_1,\ldots,x_n\right) - \int_{\{0,1\}} f\left(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_n\right) d\mu_p\left(\xi\right) \tag{1.2.7}$$

for $i = 1,2,\ldots,n$. Talagrand [90] proved the following theorem.

Theorem 1.2.10 (Theorem 1.5 of [90])**.** *There is a constant $K > 0$ such that for all $p \in (0,1)$ and $f \in L_{\mu_p}\left(\{0,1\}^n\right)$ with $\int f d\mu_p = 0$, we have*

$$\|f\|_2^2 \le K \log\left(\frac{2}{p\left(1-p\right)}\right) \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(e\left\|\Delta_i f\right\|_2 / \left\|\Delta_i f\right\|_1\right)}. \tag{1.2.8}$$

Remark 1.2.11. (i) Compared to Theorem 1.2.6, the constant on the right hand side of (1.2.8) depends on p. Moreover, as we will see in Chapter 2 or alternatively in [90], it cannot be replaced by a universal constant.

(ii) When we consider the special case where $f = \mathbf{1}_A$ we get that

$$\begin{aligned}\|\Delta_i f\|_2^2 &= p\left(1-p\right) I_A\left(i,p\right),\\ \|\Delta_i f\|_1 &= 2p\left(1-p\right) I_A\left(i,p\right).\end{aligned}$$

Substituting this to (1.2.8) we get a result slightly stronger than Theorem 1.2.6. However, as we see below, the generalization of (1.2.8) to products of probability spaces is a slightly more elaborate.

Let us motivate the proof of Theorem 1.2.10. Talagrand [90] used the following orthonormal basis of L_{μ_p}. It is analogous to the Walsh basis (1.2.1).

$$r_y\left(x\right) = \left(\frac{p}{1-p}\right)^{n/2} \prod_{i=1}^{n} \left(\frac{p-1}{p}\right)^{x_i y_i}.$$

It was constructed such that the Fourier coefficients of f and $\Delta_i f$ have a similar correspondence as those of $\mathbf{1}_A$ (1.2.2) and $\mathbf{1}_{\delta_i A}$ (1.2.3): We have

$$\Delta_i f = \sum_{y\in\{0,1\}^n :\, y_i = 1} \hat{f}\left(y\right) r_y$$

where

$$f = \sum_{y\in\{0,1\}^n} \hat{f}\left(y\right) r_y.$$

From this point, the proof of (1.2.8) is similar to the proof of Theorem 1.2.6 as described in Section 1.2.2. However, there is a crucial difference: the Bonami-Beckner inequality (Theorem 1.2.9) is not applicable since it only holds for the case $p = 1/2$. To circumvent this problem, Talagrand [90], used a symmetrization procedure, Lemma 2.1 of [90]. From this lemma, with some calculations, Theorem 1.2.10 follows.

1.2.4 Our results

In Chapter 2 we generalize Theorem 1.2.10 to the case where the underlying probability space is a product of finite probability spaces. We follow the strategy of Talagrand: we find a suitable basis, and we extend the symmetrization procedure above. Apart from the strategy of Talagrand, there is another way to extend Lemma 2.1 of [90]: Wolff in [99] computed the optimal hypercontractivity constants for operators similar to S in Theorem 1.2.9. He extended the Bonami-Beckner inequality (Theorem 1.2.9) to the case where the underlying probability space is a product of finite spaces. Using this result one can give an alternative proof of our extension of Talagrand's inequality.

Since the first appearance of our results, Cordero-Erausquin and Ledoux in [30] proved a result more general than ours. To briefly describe these results, we need some more preparation, which leads us to the next section.

Before doing so, let us briefly mention that our motivation to study the subject came from the paper [22]. There the authors, among other things, considered certain extensions of Talagrand's inequality and sharp threshold results. However, they did not prove nor state our generalization of the inequality, which would have been useful for their purposes.

1.2.5 Hypercontractivity and logarithmic Sobolev inequalities

Here we only consider the case where $p = 1/2$, however the results of this section are valid in a more general set-up as explained below. In (1.2.6) we defined the operator $T_{\varepsilon,n}$ by its action on the Walsh basis. Here we give a more natural representation of $T_{\varepsilon,n}$. First we consider the case $n = 1$. We represent a function on $\{0,1\}$ by a vector, simple computations show that $T_{\varepsilon,1}$ has the matrix form

$$T_{\varepsilon,1} = \frac{1}{2}\begin{pmatrix} 1+\varepsilon & 1-\varepsilon \\ 1-\varepsilon & 1+\varepsilon \end{pmatrix}.$$

Moreover, from the product structure of the probability space $\left(\{0,1\}^n, \mu_p^n\right)$, it follows that $T_{\varepsilon,n} = T_{\varepsilon,1}^{\otimes n}$, the n-fold tensor product of $T_{\varepsilon,1}$ by itself.

Let us consider a continuous time simple random walk on $\{0,1\}$. It has transition matrix

$$Q_1 := \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

and its stationary distribution is $\mu_{1/2}$. Let $H_{t,1}$ denote the Markov semigroup corresponding to Q_1. Simple computations show that it has the following form:

$$H_{t,1} := \exp(tQ_1) = \frac{1}{2}\begin{pmatrix} 1+e^{-2t} & 1-e^{-2t} \\ 1-e^{-2t} & 1+e^{-2t} \end{pmatrix} = T_{e^{-2t},1}.$$

The transition matrix for the continuous time simple random walk on $\{0,1\}^n$ is

$$Q_n = \sum_{i=1}^{n} \underbrace{Id \otimes \ldots \otimes Id}_{i-1} \otimes Q_1 \otimes \underbrace{Id \otimes \ldots \otimes Id}_{n-i}$$

where Id denotes the two by two identity matrix. Further simple computations shows that, $H_{t,n} := \exp(tQ_n)$, the semigroup corresponding to the continuous time simple random walk on $\{0,1\}^n$ has the form $H_{t,n} = H_{t,1}^{\otimes n}$. This allows us to restate the Bonami-Beckner inequality (Theorem 1.2.9):

Theorem 1.2.12. *Let $t \geq 0$ and $n \in \mathbb{N}$. Let $H_{t,n}$ denote the semigroup corresponding to the continuous time simple random walk on $\{0,1\}^n$. Then*

$$\|H_{t,n}(f)\|_2 \leq \|f\|_q \tag{1.2.9}$$

for $f \in L$ where $q = 1 + e^{-4t}$.

Let us turn to a more general context. Let $(\Omega, \mathcal{F}, \mu)$ be a *finite* probability space where $|\Omega| < \infty$. We define the entropy by

$$Ent_\mu(f) := \int f \log(f)\, d\mu - \int f d\mu \log\left(\int f d\mu\right). \tag{1.2.10}$$

Let Q be the transition matrix of a continuous time irreducible Markov chain on Ω which has μ as its stationary measure. Let $\mathscr{E}$ denote the Dirichlet form defined as

$$\mathscr{E}(f,g) := \int f(-Qg)\, d\mu \tag{1.2.11}$$

for $f, g \in L_\mu = L_\mu(\Omega)$. We say that the logarithmic Sobolev inequality is satisfied with a constant c if

$$cEnt\left(f^2\right) \leq \mathscr{E}(f,f) \tag{1.2.12}$$

holds for all $f \in L_\mu(\Omega)$. The logarithmic Sobolev constant ρ is the largest constant for which (1.2.12) holds:

$$\rho := \min\left\{\frac{\mathscr{E}(f,f)}{Ent(f^2)} : f \in L_\mu(\Omega),\, f \neq 0\right\}. \tag{1.2.13}$$

The following well-known result makes a connection between the hypercontractivity of $H_t := \exp(tQ)$ and the logarithmic Sobolev inequality (1.2.12). See [36] for more details.

Theorem 1.2.13 (Theorem 3.5 of [36])**.** *Let Q be a transition matrix of a finite reversible Markov chain on Ω with stationary distribution μ and logarithmic Sobolev constant ρ.*

1. *Then*
$$\|H_t f\|_2 \leq \|f\|_q$$
holds for all $t > 0$ and $2 \leq q < \infty$ satisfying $q \geq 1 + e^{-4\rho t}$.

2. *Conversely, assume that there is* $\beta > 0$ *such that*

$$\|H_t f\|_2 \leq \|f\|_q$$

for all $t > 0$ *and* $2 \leq q < \infty$ *satisfying* $q \geq 1 + e^{-4\beta t}$. *Then* (1.2.12) *holds with* $c = \beta$. *In particular,* $\rho \geq \beta$.

In view of Theorem 1.2.13, the Bonami-Beckner inequality (Theorem 1.2.9) boils down to the inequality $\rho_n \geq 1$, where ρ_n denotes the logarithmic Sobolev constant of the continuous time random walk on $\{0,1\}^n$ with transition matrix Q_n.

Recall that the Bonami-Beckner inequality (Theorem 1.2.9) was one of the main tools used in deriving Theorem 1.2.6 and Theorem 1.2.10. In view of Theorem 1.2.13, it is not surprising (though far from trivial) that one can extend Theorem 1.2.10 using logarithmic Sobolev inequalities, assuming that the logarithmic Sobolev constants of the corresponding Markov chains (or, in general, Markov processes) is non-zero. This was done by Cordero-Erausquin and Ledoux in [30]. In Theorem A.1 of [36] the logarithmic Sobolev constants are computed for some special finite Markov chains. This combined with the results in [30] give an extension of Theorem 1.2.10 and explain the constant on the right hand side of (1.2.8).

1.3 First passage percolation

The first passage percolation can be viewed as a refined version of the percolation model: we take the time for the water to pass through the tubes into account. In Section 1.3.1 we consider the model where these passage times are independent. We precisely define the model and describe some results on it such as the shape theorem and bounds on the variance of the travel time. In Section 1.3.2 we informally describe our results which are generalization of the variance results of Section 1.3.1 to the case where the passage times are 'weakly dependent'. The reader might find that the precise statement of our results and conditions of Chapter 3 are a bit technical and even a slightly mysterious at the first sight. We consider, in Section 1.3.3, the subcritical ($\beta < \beta_c$) Ising model and show that it satisfies the conditions of our general results. We achieve this in three steps. First we define the model, then we give a perfect sampling method for it. This leads us to a coding of the subcritical Ising model by i.i.d random variables, satisfying certain conditions. These conditions are of similar flavour and stronger than those required by our results in Chapter 3.

1.3.1 Independent first passage percolation

Hammersley and Welsh in [53] introduced the first passage percolation model to describe disease spreading in an orchard. To emphasize its connection to the percolation model, we give a different interpretation. Recall that in the percolation model, the porous medium was modelled as a regular structure of tubes, where each tube (edge) is open (i.e wide enough to let the water through) with probability p, and closed with probability $1-p$. We refine the model by sampling a random variable $t(e) \in [0, \infty]$ independently with identical distribution for each edge e. The passage time $t(e)$ denotes the time which is needed for the water to pass through the tube. As in the percolation model, we choose the underlying regular structure to be $\mathbb{Z}^d$ for some $d \geq 1$. The interesting question is how does the water propagate as time passes when we supply it at one given vertex. Note that the case where $t(e) = \infty$ is equivalent to saying that the edge e is closed. Hence, when we look at the set of edges which eventually became wet, we get the cluster of the origin in the ordinary percolation model with parameter $\mathbb{P}(t(e) < \infty)$.

For a path π, the passage time of π is defined as

$$T(\pi) := \sum_{e \in \pi} t(e).$$

Since the water propagates through all possible edges simultaneously, it is not hard to check that the travel time

$$T(u, v) := \inf_{\pi : u \to v} \sum_{e \in \pi} t(e) \tag{1.3.1}$$

is the time when the water reaches the vertex v when we supply it at u. In (1.3.1) the infimum is taken over the paths π which start at u and end at v. We denote the set of vertices which are wet at time $s \geq 0$ by

$$W(s) := \left\{ v \in \mathbb{Z}^d \mid T(0, v) \leq s \right\}.$$

The main object of study is $W(s)$. We are particularly interested in how fast it increases as $s \to \infty$. Before we answer this question, we investigate how $W(s)$ grows in a single direction. For $x \in \mathbb{Z}^d$ we define

$$a_{n,x} = a_n := \mathbb{E}T(0, nx).$$

The definition (1.3.1) gives that

$$T(0, (n+m)x) \leq T(0, nx) + T(nx, (n+m)x),$$

from which it follows that

$$a_{n+m} \leq a_n + a_m.$$

Thus the function $n \to a_n$ is sub-additive and the limit

$$\mu = \mu(x) = \lim_{n \to \infty} \frac{a_n}{n}$$

exists. Hence, roughly speaking, the set $W(s)$ increases in the direction x with speed $1/\mu$.

The results above raise the question if the set $W(s)/s$ has some kind of limit as $s \to \infty$. Since the set $W(s)$ is discrete, it is more convenient to put cubes around each vertex of it, and examine the resulting body. Let $U := \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ and let $\|\,.\,\|$ denote the Euclidean norm in $\mathbb{R}^d$. The following, so called shape theorem, appeared in Kesten [65]. It is essentially the same as the results of Cox and Durrett [32]:

Theorem 1.3.1 (Theorem B of [65])**.** *Assume that* $\mathbb{P}(t(e)=0)=0$, *and that*

$$\mathbb{E}\left(\min\left\{t(e_1)^d, t(e_2)^d, \ldots, t(e_{2d})^d\right\}\right) = d\int_0^\infty (1-F(z))^{2d} z^{d-1} < \infty, \tag{1.3.2}$$

where $e_1, e_2, \ldots, e_{2d}$ *are different edges of* $\mathbb{Z}^d$ *and* F *denotes the distribution function of* $t(e)$ *for some edge* e*. Then there exists a non-random set* $W_0 = W_0(F,d) \subset \mathbb{R}^d$ *such that it has a non-empty interior, compact, and is either compact or equal to all* $\mathbb{R}$*. Moreover,*

- *if* W_0 *is compact, then for all* $\varepsilon > 0$

$$(1-\varepsilon) W_0 \subset \frac{1}{s}(W(s)+U) \subset (1-\varepsilon) W_0 \text{ eventually with probability } 1,$$

- *if* $W_0 = \mathbb{R}^d$*, then for all* $\varepsilon > 0$

$$\left\{x \in \mathbb{R}^d \,\middle|\, \|x\| \leq \varepsilon^{-1}\right\} \subset \frac{1}{s}(W(s)+U) \text{ eventually, with probability } 1.$$

In addition, W_0 *is invariant under permutations of the coordinates or reflections in the coordinate system.*

If (1.3.2) *fails, then, almost surely,*

$$\limsup_{v\to\infty} \frac{1}{\|v\|} T(0,v) = \infty.$$

Theorem 1.3.1 above gives an almost complete result on the existence of the limiting shape of $\frac{1}{s}(W(s)+U)$ in the case where the passage times of the vertices are independent. In particular, under (1.3.2), we have that

$$\frac{1}{n} T(0,nx) \to \mu \text{ a.s.} \tag{1.3.3}$$

The results above do not give any bounds on the speed of convergence in (1.3.3). Nonetheless, the speed of convergence is an interesting question on its own. Moreover, this question places the first passage percolation to a broader perspective: The first passage percolation is conjectured to be in the Kardar-Parisi-Zhang (KPZ) universality class [63]. See [31] for a recent review for KPZ, and [29] for some recent results related to (1.3.4) below. If indeed this conjecture is true, then it implies that for $d=2$,

$$Var(T(0,nx)) \asymp n^{2/3}. \tag{1.3.4}$$

Unfortunately, no result close to (1.3.4) is known up to now. To our knowledge, the best known general result is due to Kesten:

Theorem 1.3.2 (Theorem 1 of [65])**.** *Consider the first passage percolation on* $\mathbb{Z}^d$ *for some* $d \in \mathbb{N}$. *Suppose that*

$$Var\,(t\,(e)) < \infty \text{ and } \mathbb{P}\,(t\,(e) = 0) < p_c\left(\mathbb{Z}^d\right) \tag{1.3.5}$$

holds, where $p_c\left(\mathbb{Z}^d\right)$ *denotes the critical parameter for the bond percolation on* $\mathbb{Z}^d$. *Then there is* $c = c\,(x, F, d)$ *such that*

$$Var\,(T\,(0, nx)) \leq cn \tag{1.3.6}$$

for all $n \in \mathbb{N}$.

Benjamini, Kalai and Schramm [13] improved the upper bound in (1.3.6) for the case where $t\,(e)$ can take two different positive values a and b each with probability $1/2$. They showed that there is $c = c\,(a, b, d)$ such that

$$Var\,(T\,(0, v)) \leq c\frac{|v|}{\log |v|} \tag{1.3.7}$$

for $v \in \mathbb{Z}^d$ with $|v| \geq 2$. In the derivation of (1.3.7), the inequality (1.2.8) due to Talagrand was used. Later Benaim and Rossignol [12] used logarithmic Sobolev inequalities similar to (1.2.12) to deduce the bound (1.3.7) for a large class of passage time distributions.

1.3.2 Our results

We generalize the results above to a different direction. In Chapter 3 we show that the bound (1.3.7) holds in the case where the (random) passage times $\left\{t\,(e) \,\middle|\, e \in \mathbb{Z}^d\right\}$ are translation invariant, 'weakly dependent' and take values in the interval $[a, b]$ for some $b > a > 0$ and $d \geq 2$.

We follow the strategy of Benjamini, Kalai and Schramm [13]. We use Theorem 2.1.3, a general version of Talagrand's inequality. The main technical difficulty is that this inequality holds only for i.i.d random variables, hence we cannot use it directly to the passage times $t\,(e)$ $e \in \mathbb{Z}^d$ in the weakly dependent case. The key idea is to use Theorem 2.1.3 to the i.i.d random variables which *encode* the passage times. However, this encoding has to satisfy some conditions to make the arguments work. This brings us to the next section.

1.3.3 Key example: the Ising model

We consider the site version of the first passage percolation model on $\mathbb{Z}^d$ for $d \geq 2$, where the vertices have passage times instead of the edges. We show that our results of Chapter 3 hold for the Ising model, where we replace the $+1$ (-1) spins with a (b) for some $a, b > 0$. Since existence of the collection of coding variables above do not depend on the values of a and b, we keep the ± 1 spins in the following.

Definition and phase transition

The Ising model was introduced by Lenz [71] for ferromagnetism. It was first studied by Ising [59]. Similarly to the percolation model, the Ising model can be defined for any graph G. In this short description we restrict to the case where this graph is $\mathbb{Z}^d$ for $d \geq 2$. Moreover, we only consider the model without external field, and the interaction parameter is constant on the edges. The general definition of the Ising model can be found in any standard textbook on statistical mechanics see [47] for example.

For $S \subseteq \mathbb{Z}^d$ let ∂S denote the outer boundary of S, that is

$$\partial S = \left\{ u \in \mathbb{Z}^d \setminus S \mid \exists v \in S \text{ s.t. } u \sim v \right\}$$

where $u \sim v$ denotes that u and v are neighbouring vertices in $\mathbb{Z}^d$. Let $n \in \mathbb{N}$ and recall the definition of $B(n)$ from (1.1.4). A spin can have value either $+1$ or -1, that is we consider configurations $\eta \in \{-1,+1\}^{\mathbb{Z}^d}$. We introduce the partial order $\leq$ on $\{-1,+1\}^{\mathbb{Z}^d}$ as

$$\eta \leq \eta' \Leftrightarrow \forall v \in \mathbb{Z}^d \, \eta_v \leq \eta'_v$$

for $\eta, \eta' \in \{-1,+1\}^{\mathbb{Z}^d}$. For $\eta \in \{-1,+1\}^{\mathbb{Z}^d}$ and $n \in \mathbb{N}$ the Hamiltonian H_n^η is defined as

$$H_n^\eta(\sigma) := -\sum_{u,v \in B(n):\, u \sim v} \sigma_u \sigma_v - \sum_{u \in B(n), v \in \partial B(n):\, u \sim v} \sigma_u \eta_v \tag{1.3.8}$$

for a configuration $\sigma \in \{-1,+1\}^{B(n)}$. Let $\beta > 0$ be a parameter called the inverse temperature. We define the probability measure $\mu_{n,\beta}^\eta$ on $\{-1,+1\}^{B(n)}$ as follows:

$$\mu_{n,\beta}^\eta(\{\sigma\}) := \frac{1}{Z} \exp\left(-\beta H_n^\eta(\sigma)\right) \tag{1.3.9}$$

for $\sigma \in \{-1,+1\}^{B(n)}$ where $Z = Z(n,\beta,\eta)$ is a normalizing constant (called the partition function). In the case where η is constant $+1$ (-1) we use $+$ $(-)$ instead of η in the notation above. It is a simple exercise to show that for all $\eta \in \{-1,+1\}^{\mathbb{Z}^d}$ the measure $\mu_{n,\beta}^\eta$ is 'between' the measures $\mu_{n,\beta}^+$ and $\mu_{n,\beta}^-$. That is, $\mu_{n,\beta}^\eta$ stochastically dominates $\mu_{n,\beta}^-$, and it is stochastically dominated by $\mu_{n,\beta}^+$.

Remark 1.3.3. Note that for $\beta = 0$, (1.3.9) says that, independently of the choice of $\eta \in \{-1,+1\}^{\mathbb{Z}^d}$ all the configurations $\sigma \in \{-1,+1\}^{B(n)}$ have the same probability. Hence each vertex has spin $+1$ (-1) with probability $1/2$ independently from each other. Thus the vertices with spin $+1$ have the same distribution as the open vertices in site percolation with parameter $1/2$ in $B(n)$.

For any fixed $\beta > 0, \eta \in \{-1,+1\}^{\mathbb{Z}^d}$ and cylinder event $A \subset \{-1,+1\}^{\mathbb{Z}^d}$ the sequence $\mu_{n,\beta}^\eta(A)$ has a subsequential limit as $n \to \infty$. The diagonal argument combined with general results from measure theory give that there is a sequence

$(n_k)_{k\in\mathbb{N}}$ and a measure μ_β^η such that for *any* cylinder event A in $\{-1,+1\}^{\mathbb{Z}^d}$ we have

$$\mu_\beta^\eta(A) = \lim_{k\to\infty} \mu_{n_k,\beta}^\eta(A). \tag{1.3.10}$$

Note that the measure μ_β^η might be different for different choices of η and of the sequence $(n_k)_{k\in\mathbb{N}}$. One way to characterize the different phases of the Ising model is through the number of possible measures μ_β^η. Let μ_β^+ (μ_β^-) denote one of the limiting measures in (1.3.10) where η is constant $+1$ (-1). From the remark below (1.3.9) we get that there is a unique measure satisfying (1.3.10) if and only if for all possible choices of the limiting measures μ_β^+ and μ_β^- we have $\mu_\beta^+ = \mu_\beta^-$. It is well known that for $d \geq 2$ there is $\beta_c = \beta_c(d) > 0$ such the measure in (1.3.10) is unique for $\beta < \beta_c$, while it is non-unique for $\beta > \beta_c$. See [47] for more details.

Exact simulation by coupling from the past

Here we restrict to the high temperature ($\beta < \beta_c$) case. Hence the measure $\mu_\beta = \mu_\beta^\eta$ is unique, and the limit in (1.3.10) can be extended from the subsequence n_k to the full sequence n.

The special form of the Hamiltonian (1.3.8) and the measure μ_β implies that the Ising model is a Markov random field: Given the spins on $\mathbb{Z}^d \setminus \{v\}$ the (conditional) distribution of the spin at vertex v is the same as if we would only condition on the spins at vertices neighbouring v. In particular, for all $\alpha \in \{-1,+1\}^{\mathbb{Z}^d}$ and $v \in \mathbb{Z}^d$ we have

$$\mu_\beta(\sigma_v = +1 \,|\, \sigma_u = \alpha_u \,\forall u \neq v) = \frac{\exp(\beta \sum_{u\sim v} \alpha_u)}{\exp(\beta \sum_{u\sim v} \alpha_u) + \exp(-\beta \sum_{u\sim v} \alpha_u)}. \tag{1.3.11}$$

Using (1.3.11), we define the following updating procedure. Let $\alpha \in \{-1,+1\}^{\mathbb{Z}^d}$ and $v \in \mathbb{Z}^d$. The local update of α at v is a configuration $\alpha' \in \{-1,-1\}^{\mathbb{Z}^d}$ such that $\alpha'_u = \alpha_u$ for $u \neq v$ and

$$\alpha'_v = \begin{cases} +1 & \text{if } X \geq \mu_\beta(\sigma_v = -1 \,|\, \sigma_u = \alpha_u \,\forall u \neq v) \\ -1 & \text{otherwise,} \end{cases}$$

where the random variable X has uniform distribution on the interval $[0,1]$.

Remark. The local updates described above are called as heath-bath or Glauber dynamics in the literature.

The procedure above allows us to couple local updates at vertex v for different configurations α by using the same random value X in (1.3.3). Since the right hand side of (1.3.11) is a function of α_u for vertices $u \sim v$, hence $\mu_\beta(\sigma_v = +1 \,|\, \sigma_u = \alpha_u \,\forall u \neq v)$ can only take finitely many different values. Thus we can replace X by a suitable random variable which takes only finitely many different values and get the same coupling.

Moreover, when

$$X \geq 1 - \min_{\alpha} \mu_\beta \left(\sigma_v = +1 \,|\sigma_u = \alpha_u \,\forall u \neq v \right)$$

or when

$$X \leq \min_{\alpha} \mu_\beta \left(\sigma_v = -1 \,|\sigma_u = \alpha_u \,\forall u \neq v \right),$$

then we do not have to look at the spins of the neighbours of v, since in the first (second) case no matter the boundary condition, the spin is going to be set to $+1$ (-1). Hence with probability

$$\begin{aligned} &\gamma = \gamma\left(\beta, d\right) \\ &\quad := \min_{\alpha} \mu_\beta \left(\sigma_v = +1 \,|\sigma_u = \alpha_u \,\forall u \neq v \right) + \min_{\alpha} \mu_\beta \left(\sigma_v = -1 \,|\sigma_u = \alpha_u \,\forall u \neq v \right) \end{aligned} \tag{1.3.12}$$

we do not have to look at the spins neighbouring v to evaluate the updated value of the spin at v.

Using the local updates above, we construct a perfect simulation for sampling from the distribution $\mu^{\eta}_{n,\beta}$ as follows. Let $\alpha^{\eta,0} \in \{-1,+1\}^{B(n)}$ be a starting configuration, and $\eta \in \{-1,+1\}^{\mathbb{Z}^d}$. At round 1, we start with configuration $\alpha^{\eta,0}$ in $B\left(n\right)$. Each round, we update the spins in a chessboard fashion: We say that a vertex $v = (v_1, \ldots, v_d) \in \mathbb{Z}^d$ is odd (even) if $v_1 + \ldots + v_d$ is odd (even). In odd (even) rounds, we update at odd (even) vertices of $B\left(n\right)$. At round k, we get the configuration $\alpha^{\eta,k} = \alpha^{\eta,k}\left(\beta, d, \underline{X}\right)$ from $\alpha^{\eta,k-1}$ by the update rule described above. Here $\underline{X}$ refers to the set of all the random variables $X_{v,t}$ which is the source of randomness for the local update at vertex $v \in B\left(n\right)$ at round $t \in \{1, 2, \ldots, k\}$.

Note that the process $k \to \alpha^{\eta,k}$ is a discrete time Markov chain. Let us use the same source of randomness $\underline{X}$ and the same η, but with different starting configurations $\alpha^{\eta,0}$. We arrive to the so called grand coupling of the Markov chain above. (See the definition of grand coupling in Section 5.4 of [72]). Let $\alpha^{\eta,k}_{+}$ $(\alpha^{\eta,k}_{-})$ denote the configuration above when we start with $\alpha^{\eta,0}$ being constant $+1$ (-1). Let

$$\tau = 1 + \inf\left\{ k \in \mathbb{N} \,\middle|\, \alpha^{\eta,k}_{+} = \alpha^{\eta,k}_{-} \right\}.$$

It is easy to check that τ is a stopping time with respect to the natural filtration of $\underline{X}$. Moreover, for the coupling above, we have $\alpha^{\eta,k}_{-} \leq \alpha^{\eta,k} \leq \alpha^{\eta,k}_{+}$ for any starting configuration $\alpha^{\eta,0}$. This implies that $\alpha^{\eta,\tau}$ has distribution $\mu^{\eta}_{n,\beta}$. (In particular, τ is a strong stationary time as defined in Section 6.4 of [72].) This gives a perfect sampling method for $\mu^{\eta}_{n,\beta}$, which we use in the following.

A coding of the Ising model in terms of i.i.d random variables

We start with a spin configuration $\sigma \in \{-1,+1\}^{\mathbb{Z}^d}$ which has distribution μ_β. The key idea is to look at the spin σ_v as if it would be a result of a perfect sampling procedure of the above. Then we ask the question how many rounds do we have to go back in time so that we would already know the value of σ_v by only looking at the update variables $X_{u,t}$ for $u \in \mathbb{Z}^d$ and $t \leq 0$. Clearly, this question is related to the mixing time of the Markov chains above. In the following, roughly speaking, we bound this mixing time by looking at a certain space-time diagram where we compare the set of the update variables which affect the value of σ_v with sub-critical Galton-Watson trees. The arguments below appeared in the proof of Proposition 2.1 of Häggström and Steif [52].

Let I denote the set of pairs (u,t) such that $u \in \mathbb{Z}^d$ and $-t \in \mathbb{N}$ with $t + \sum_{i=1}^d u_i$ is even. Let us take i.i.d random variables $X_{u,t} \sim Unif[0,1]$ for $(u,t) \in I$. We put a directed edge starting from (u,t) and pointing to $(u', t-1)$ for all $(u,t), (u', t-1) \in I$ such that $\|u - u'\|_1 = 1$.

We say that $(u,t) \in I$ is bad if

$$X_{u,t} \in \Big(\min_{\alpha} \mu_\beta \left(\sigma_v = -1 \,|\sigma_u = \alpha_u \,\forall u \neq v \,\right), \max_{\alpha} \mu_\beta \left(\sigma_v = -1 \,|\sigma_u = \alpha_u \,\forall u \neq v \,\right) \Big).$$

By (1.3.12) (u,t) is bad with probability $1-\gamma$. For $(u,t) \in I$ let $CB(u,t)$ denote the directed cluster of (u,t) in the induced subgraph of the bad vertices of I. It is the set of elements of I which are reachable by a directed path from (u,t). Furthermore, let $h(u,t)$ denote the height of $CB(u,t)$ for $(u,t) \in I$, that is

$$h(u,t) := 1 + \sup \left\{ |s| \,\middle|\, \exists w \in \mathbb{Z}^d \text{ such that } (w,s) \in CB(u,t) \right\}.$$

It is easy to check that there is a constant $c_1 = c_1(d)$ such that

$$CB(u,t) \leq c_1 h(u,t)^{d+1}. \tag{1.3.13}$$

With a slight abuse of notation, we define

$$CB(v) := \begin{cases} CB(v,0) & \text{if } v \text{ is even} \\ CB(v,-1) & \text{if } v \text{ is odd.} \end{cases}$$

We define $h(v)$ analogously. By comparing the $CB(v)$ with Galton-Watson trees, standard theory of branching processes (see Theorem 1 on page 40 of [7]) gives that under the so called high-noise (HN) condition (see [51, 52])

$$\gamma > \frac{2d-1}{2d}, \tag{1.3.14}$$

we have

$$\mathbb{P}(h(v) > k) \leq \exp(-c_2 k)$$

for some $c_2 = c_2(\gamma, d)$ and for all $k \in \mathbb{N}$. In particular, we get that $h(v)$ is a.s. finite. Note that the HN condition (1.3.14) for the Ising model is satisfied for $0 \leq \beta < \beta_{HN}(d)$, for some $0 < \beta_{HN}(d) < \beta_c(d)$.

Let $\alpha \in \{-1,+1\}^{\mathbb{Z}^d}$ and $v \in \mathbb{Z}^d$. Let us use the aforementioned updating procedure starting at time $-h(v)$ with some starting configuration α using the update variables $X_{u,t}$ $(u,t) \in I$ with $t > -h(v)$. After $h(v)$ rounds we get the random configuration $\alpha^{h(v)}$. By the definition of $h(v)$, the value of $\alpha_v^{h(v)}$ only depends on the update variables $X_{u,t}$ for $(u,t) \in CB(v)$, but not on the configuration α. Hence $\sigma_v := \alpha_v^{h(v)}$ is well defined and it is a function of the update variables $X_{u,t}$ $(u,t) \in I$. The results imply that σ has distribution μ_β under the HN condition (1.3.14). Moreover, σ is a function of $X_{u,t}$ $(u,t) \in I$ with the following properties:

- There exists a constant $c_3 = c_3(\gamma, d)$ such that for each $v \in \mathbb{Z}^d$ there is a sequence $i_1(v), i_2(v), \ldots$ of elements of I such that for all $k \in \mathbb{N}$,

$$\mathbb{P}\big(\big(X_{i_1(v)}, X_{i_2(v)}, \ldots, X_{i_k(k)}\big) \text{ does not determine } \sigma_v\big) \leq \exp\left(-c_3 k^{1/(d+1)}\right). \tag{1.3.15}$$

- $\exists c_4 > 0$ such that $\forall u, v \in \mathbb{Z}^d$ and $\forall k < c_4 |u-v|^{d+1}$,

$$\{i_1(v), i_2(v), \ldots, i_k(v)\} \cap \{i_1(u), i_2(u), \ldots, i_k(u)\} = \emptyset. \tag{1.3.16}$$

- The distribution of σ is translation invariant.

The properties above are similar and stronger to those required by our results in Chapter 3: instead of the exponential bound in (1.3.15), it is enough to have a polynomial upper bound of order $k^{-3d-\varepsilon}$ for some $\varepsilon > 0$. Moreover, it is enough that (1.3.16) holds for $k < c_5 |u-v|$ where c_5 is some positive constant. See Conditions (i)-(iii) of Chapter 3.

In this section we gave a sketch-proof for the fact that the properties above hold for the Ising model under the condition for $d \geq 2$ and $0 \leq \beta < \beta_{HN}(d)$. In Chapter 3 we extend this result by showing that these properties hold for the Ising model for $d \geq 2$ and $0 \leq \beta < \beta_c(d)$.

1.4 Frozen percolation

Let $G = (V, E)$ be a graph. Recall Section 1.1 where we defined a coupling between percolation models with different parameters. We can interpret this coupling as a growth process where the edge $e \in E$ opens at time τ_e. In particular, when we consider the evolution of the open clusters, we get a process where clusters merge over time and at time 1 there is only one clusters the whole graph G.

In the following we modify the process above so that 'big' clusters do not interact with the other clusters – they freeze. We achieve this by opening the edge e at time τ_e if and only if the open clusters of the endpoints of e have size less than N, where $N \in \mathbb{N}$ is a parameter of the model. The motivation for this process comes from a polymerization model of Stockmayer [89]. For more background see Section 5.1.

As in Section 1.1, we are interested in the large N behaviour of the model. A natural way to approach this question is to consider the so called ∞-parameter model where open clusters freeze as soon as they become infinite. However, this approach has some limitations: as we see below, the ∞-parameter process might not exist, while the N-parameter processes do exist under some weak conditions. Aldous in [4] gave a construction for the ∞-parameter process on the planted binary tree. We briefly sketch this construction and some of the results of [4] in Section 1.4.1. In Section 1.4.2 we motivate the results in Chapter 4 where we show that the N-parameter processes converge to Aldous' process. In Section 1.4.3 we turn to the ∞-parameter model on the square lattice. We outline the proof of a result by Benjamini and Schramm [14] which states that this process does not exist. Hence the approach above fails in this case. Thus we turn to the methods, similar to those in Section 1.1.3, of critical and near-critical percolation. Using these methods we investigate the N-parameter processes on the square and triangular lattice in Chapter 5. We finish the introduction with Section 1.4.4 where we motive our results in Chapter 5.

1.4.1 The model of Aldous, and some of his results

Aldous [4] considered the ∞-parameter process on the infinite binary and infinite planted binary tree. Since the methods and the results are quite similar for the normal and for the planted binary tree, we only consider the latter case. Let $T = (V, E)$ denote the planted binary tree. It is the infinite tree where the root vertex v_0 has degree 1 while all the other vertices have degree 3. For an edge $e \in E$, let $T_e = (V_e, E_e)$ denote the following subgraph of T. E_e is the set of edges e' such that the path connecting v_0 with e' contains e. V_e is the set of the endpoints of the edges of E_e. Note that T_e is also a planted binary tree for all $e \in E$. This recursive property of T plays a central role in the arguments below. Similarly to the strategy employed in [4, 5], we first assume that the ∞-parameter process exists, and find some of its properties which in turn help us to construct the process.

For $e \in E$, let $Y_e \geq 0$ be the random time when the open cluster of e becomes infinite (i.e e freezes) in the ∞-parameter frozen percolation process on T_e. If the edge e never freezes in this process, then we set $Y_e = \infty$. Let e_0 denote the edge of the root, and e_1, e_2 denote the edges which share an endpoint with e_0. It is easy to check that

$$Y_{e_0} = \begin{cases} \infty & \text{if } Y_{e_1} \wedge Y_{e_2} \leq \tau_{e_0} \\ Y_{e_1} \wedge Y_{e_2} & \text{if } Y_{e_1} \wedge Y_{e_2} > \tau_{e_0}. \end{cases} \tag{1.4.1}$$

The graphs T_{e_1} and T_{e_2} have no common edges, hence it is natural to assume that Y_{e_1}, Y_{e_2} and τ_{e_0} are independent, and Y_{e_1}, Y_{e_2} have the same distribution. Under these assumptions, (1.4.1) provides a fixed-point equation for the distribution of Y_{e_0}. As it turns out, there are some simple solutions for this equation:

Lemma 1.4.1 (Lemma 3 of [4]). *Let μ be a law on $\left[\frac{1}{2}, 1\right] \cup \{\infty\}$, which is non-atomic on $\left[\frac{1}{2}, 1\right]$. Then μ satisfies the fixed-point equation above, if and only if there is $x_0 \in \left[\frac{1}{2}, 1\right]$ such that*

$$\mu(dx) = \frac{1}{2x^2} dx, \ \textit{for } x \in \left[\frac{1}{2}, x_0\right]; \ \mu(\infty) = \frac{1}{2x_0}.$$

One might wonder why we only considered solutions which are concentrated on $\left[\frac{1}{2},1\right]\cup\{\infty\}$ in Lemma 1.4.1. The reason comes from the percolation model on T. When we forget about freezing (i.e. each edge e opens at time τ_e no matter the sizes of the neighbouring clusters) we get the usual coupling of percolation models. Hence no cluster can freeze before time $p_c(T)$, where $p_c(T)$ is the critical parameter for the percolation model on T. Basic results from branching processes imply that $p_c(T)=1/2$.

Let us turn to the construction of the ∞-parameter process. Among the distributions of Lemma 1.4.1, let ν denote the distribution where $x_0=1$. For an edge $e_0'\in E$, let e_1',e_2' denote the edges which share an endpoint with e_0' in $T_{e_0'}$. Combination of the Kolmogorov extension theorem and the recursion (1.4.1), one can easily construct an infinite collection of random variables (τ_e,Y_e) $e\in E$ which satisfy the following conditions

1. The recursion (1.4.1) is satisfied not only for the edge e_0, but also for all $e_0'\in E$ when we replace the edges e_i by e_i' for $i=0,1,2$.
2. Y_e has distribution ν for all $e\in E$.

From this collection of random variables we can define the following process. At time 0 all the edges are closed. Then an edge e becomes open at time $t=\tau_e$ if for all edges e' which share a vertex with e, we have $Y_{e'}>\tau_e$. It is easy to check that the process defined above indeed satisfies the definition of the ∞-parameter frozen percolation process. Hence we constructed an ∞-parameter process on the planted binary tree.

Note that in the argument above, for any distribution μ of Lemma 1.4.1, we can construct a collection of random variables which satisfy Condition 1 and the distribution of Y_e is μ for all $e\in E$. However, it is easy to check that the resulting process on T satisfies the definition of the ∞-parameter frozen percolation process only in the case where $\mu=\nu$. Nevertheless, the arguments above do not imply the uniqueness of the ∞-parameter process on T. There are two different reasons: the first one is that Lemma 1.4.1 characterizes the solutions of (1.4.1) which have no atoms on $[1/2,1]$. The second reason is that we imposed the condition that Y_{e_1} and Y_{e_2} are independent when we solved (1.4.1). To our knowledge, the uniqueness of the ∞-parameter frozen percolation process on T is an open question. See [4] for more details.

From the construction above simple computations give the following.

Lemma 1.4.2. *Let $\beta_\infty(t)$ denote the probability that the edge e_0 is closed at time $t\in[0,1]$ in the ∞-parameter frozen percolation process constructed above. Then*
$$\beta_\infty(t)=\begin{cases}1-t & \text{for } t\in[0,1/2]\\ \frac{1}{4t} & \text{for } t\in[1/2,1].\end{cases}\tag{1.4.2}$$

Further computations show that at time $t\in[1/2,1]$, the finite open clusters are distributed like critical percolation clusters, while the frozen open clusters are distributed as incipient infinite clusters (IIC). The IIC is, roughly speaking a 'critical percolation cluster conditioned to be infinite'. See [67] for a precise definition of IIC for the binary tree. These results show that the ∞-parameter process has a self-organized critical (SOC) behaviour: without any tuning, the process drives itself to a state similar to the critical state of the percolation model at time $1/2$, and stays there till time 1. This SOC property of the ∞-parameter process makes it more interesting, since the SOC phenomenon is observed in the nature: examples include but not limited to earthquakes and avalanches. A common feature of these systems is their complexity, and due to their critical/near-critical properties, prediction of their future behaviour is also quite challenging. See [60] for more details.

1.4.2 Our results for the N-parameter process on the binary tree

We turn to the N-parameter frozen percolation model on the planted binary for large but finite N. In this case there is a slightly hidden additional parameter, the size function, the way we measure the size of the clusters. In Chapter 4 we show that under some natural conditions on the size function (Definition 4.1.3), the N-parameter frozen percolation processes in some weak sense converge to the ∞-parameter process.

Note that the N-parameter process exists, is unique and is a measurable function of the random variables τ_e $e \in E$. See [73] and [38]. Hence our convergence result shows that the ∞-parameter process constructed above is a natural process. Bandyopadhyay [9] together with Aldous [5] investigated the question whether the ∞-parameter process is a measurable function of the τ values. Unfortunately, despite their best efforts, this question is still open. Nonetheless, our results could also be useful in the investigation of such questions.

We use the following method to prove our results. Analogously to (1.4.2), we denote by $\beta_N(t)$ the probability that the root edge is closed at time $t \in [0,1]$ in the N-parameter process with some fixed good size function as in Definition 4.1.3. Using the recursive property of the planted binary tree described in the beginning of Section 1.4.1, we show that β_N is a solution of a certain differential equation. Then we solve this equation and get an implicit formula for β_N. Using this formula we prove that $\beta_N(t)$ converges to $\beta_\infty(t)$ as $N \to \infty$ for all $t \in [0,1]$. From this, by simple calculations, we get that the distribution of non-frozen (with size less than N) clusters in the N-parameter process tend to the distribution of finite clusters in the ∞-parameter process. See Chapter 4 for more details. In the following section we turn to the N-parameter frozen percolation process in two dimensions. We will see that these processes have a fundamentally different behaviour.

1.4.3 The non-existence of the ∞-parameter frozen percolation process in two dimensions

Benjamini and Schramm [14] showed the non-existence result in the title of this section. Unfortunately, to our knowledge, they did not publish their arguments

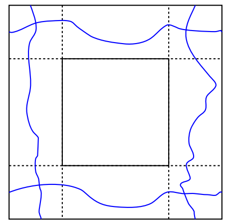

Figure 1.1: The four open crossings of the rectangles imply the occurrence of the event $M_{k,o}$.

to date. In the following we give, a possibly different, sketch proof of their result.

Let us suppose that the ∞-parameter frozen percolation process does exists on the square lattice. Our aim is to arrive to a contradiction.

Note that we can couple the percolation models with different parameters to the ∞-parameter frozen percolation process using the τ values similarly as in Section 1.1.2. This allows us to compare the percolation model with parameter $t \in [0,1]$ with the ∞-parameter frozen percolation process at time t.

Recall the notation and the results of Section 1.1. For $n, m \in \mathbb{N}$ with $m < n$ let

$$A(m,n) := B(n) \setminus B(m)$$

denote the annulus with inner radius m and outer radius n. For $k \in \mathbb{N}$ we set $A_k := A\left(2^k, 2^{k+1}\right)$. For $k \in \mathbb{N}$ let $M_{k,o}$ $(M_{k,c})$ denote the event where there is an open (closed) circuit in A_k around $B\left(2^k\right)$. Moreover, let $M^*_{k,o}, M^*_{k,c}$ denote the dual version of the events above.

Note that the crossing event $\mathcal{H}_o([0,an] \times [0,bn])$ is increasing as of Definition 1.2.3. Hence these open (closed) crossing events are positively correlated by an inequality due to Fortuin, Ginibre and Kasteleyn [41]. (See Theorem 5.2.2 for a precise formulation.) This together with Theorem 1.1.3 (RSW) imply that the probabilities $\mathbb{P}_{1/2}(M_{k,o}), \mathbb{P}_{1/2}(M^*_{k,c})$ are bounded away from 0 and 1. See Figure 1.1 for more details. This, together with the Borel-Cantelli lemma imply that with probability 1, there are infinitely many disjoint open circuits and closed dual circuits around the origin in the percolation model with parameter 1/2. This has the following consequences for the ∞-parameter frozen percolation process:

- Let F be an infinite open, i.e frozen, cluster in the ∞-parameter frozen percolation process. The existence of the open circuits above imply that infinitely many of them are contained in F. This further implies that there is a unique frozen cluster, which we denote by F.

- Let p_F denote the time when F formed. Then the existence of the closed circuits above give that $p_F > 1/2$.

We get a contradiction: let $s \in (1/2, p_F)$ be arbitrary. Then with probability 1, at time s there is an infinite open cluster in the frozen percolation process, which differs from F, which gives a contradiction with the definition of the ∞-parameter process. This finishes the proof of the non-existence of the ∞-parameter frozen percolation process. See Remark (i) on page 183 of [97] for a slightly different argument.

1.4.4 Our results on two dimensional frozen percolation

The non-existence of the ∞-parameter process on the square lattice suggests that the behaviour of the N-parameter processes for large N are quite different from that of the analogous processes on the binary tree. In particular, we will see in Chapter 5 that the large N behaviour of the N-parameter process is highly dependent on the way we measure the size of the clusters. Hence we concentrate on the case where we freeze clusters as soon as their diameter reaches N, where the diameter is the L^∞-diameter inherited from $\mathbb{R}^2$.

We show, among many other things, that the proportion of the edges which eventually freeze in the N-parameter processes on the square lattice tends to 0 as $N \to \infty$. Our results lead to the following informal description of the large N behaviour of the process. At time 0 all the vertices are closed. Then, one after the other, vertices open till time slightly before 1/2, when the first frozen clusters appear. Then more frozen clusters emerge, but starting slightly after 1/2 no more frozen clusters form. Then the vertices in the non-frozen parts open till time 1. In particular, the frozen clusters give a tiling of the plane at time 1 where the typical open clusters have diameter of order N. See the lines below Corollary 5.1.7 for more details.

We derive our results in Chapter 5 using delicate tools from near-critical percolation. For technical reasons similar to those in Section 1.1.3, in Chapter 5 we treat the site version of the N-parameter frozen percolation process on the triangular lattice, and indicate the modifications of our arguments which are required to deduce our results for the N-parameter frozen percolation process on the square lattice.

Finally, we recall from Section 1.1.3 a work in progress of Garban, Pete and Schramm [45] where the aim is to prove the scaling limit of the near-critical ensembles to a limiting process. Building on this work, we formulate a conjecture, which, roughly speaking, states that when we scale space by N, the N-parameter processes close to time 1/2 converge to a limiting process as N tends to infinity.

2 A generalization of Talagrand's inequality

This chapter is based on the paper [69].

Abstract

Let $X_1, \ldots, X_n$ be independent Bernoulli random variables and f a function on $\{0,1\}^n$. In the well-known paper [90] Talagrand gave an upper bound for the variance of f in terms of the individual influences of the X_i's. This bound turned out to be very useful, for instance in percolation theory and related fields.

In many situations a similar bound was needed for random variables taking more than two values. Generalizations of this type have indeed been obtained in the literature (see e.g. [30]), but the proofs are quite different from that in [90]. This might raise the impression that Talagrand's original method is not sufficiently robust to obtain such generalizations.

However, our paper gives an almost self-contained proof of the above mentioned generalization, by modifying step-by-step Talagrand's original proof.

Keywords and phrases. influences, concentration inequalities, sharp threshold.
AMS 2010 classifications. Primary 42B05; Secondary 60B15, 60B11.

2.1 Introduction and statement of results

2.1.1 Statement of the main results

Let $(\Omega, \mathcal{F}, \mu)$ be an arbitrary probability space. We denote its n-fold product by itself by $(\Omega^n, \mathcal{F}^n, \mu^n)$. Let $f : \Omega^n \to \mathbb{C}$ be a function with finite second moment, that is $\int_{\Omega^n} |f|^2 d\mu^n < \infty$. The influence of the ith variable on the function f is defined as

$$\Delta_i f(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - \int_\Omega f(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_n)\, \mu(d\xi)$$

for $x = (x_1, \ldots, x_n) \in \Omega^n$ and $i = 1, \ldots, n$. We will use the notation $\|f\|_q$ for the L^q norm $q \in [1, \infty)$ of f, that is $\|f\|_q = \sqrt[q]{\int_{\Omega^n} |f|^q d\mu^n}$.

Using Jensen's inequality, Efron and Stein gave the following upper bound on the variance of f (see [39]):

$$Var\,(f) \leq \sum_{i=1}^{n} \|\Delta_i f\|_2^2 \,. \tag{2.1.1}$$

In some cases (2.1.1) has been improved. We write $\mathcal{P}\,(S)$ for the power set of a set S. For the case when Ω has two elements, say 0 and 1, and $\mu\,(\{1\}) = 1 - \mu\,(\{0\}) = p$, Talagrand showed the following result:

Theorem 2.1.1 (Theorem 1.5 of [90])**.** *There exists a universal constant K such that for every $p \in (0,1)\,,\ n \in \mathbb{N}$ and for every real valued function f on $\left(\{0,1\}^n, \mathcal{P}\left(\{0,1\}^n\right), \mu_p\right)$,*

$$Var\,(f) \leq K \log\left(\frac{2}{p(1-p)}\right) \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(e\,\|\Delta_i f\|_2 \,/\, \|\Delta_i f\|_1\right)}, \tag{2.1.2}$$

where μ_p is the product measure on $\{0,1\}^n$ with parameter p.

Remark 2.1.2. An alternative proof of Theorem 2.1.1 for the case $p = 1/2$ can be found in [13].

Inequality (2.1.2) gives a bound on $Var\,(f)$ in terms of the influences. It is useful when the function f is complicated, but its influences are tractable. Such situations occur for example in percolation theory (see for example [13, 21, 93]). Further consequences of (2.1.2) include for example the widely used KKL lower bound for influences [62] and various so called sharp-threshold results e.g. [42].

In some cases, a generalization of Theorem 2.1.1 to the case $\{0, 1, \ldots, k\}^n$ with $k > 1$ is useful, for example in [22, 35]. However, up to our knowledge, no such generalization has been explicitly stated in the literature. The main goal of our paper is to present and prove an explicit generalization, Theorem 2.1.3 below. We have used this theorem and referred to it in [95].

Theorem 2.1.3. *There is a universal constant $K > 0$ such that for each finite set Ω each measure μ on Ω with $p_{min} = \min_{j \in \Omega} \mu\,(\{j\}) > 0$, and for all complex valued functions f on $(\Omega^n, \mathcal{P}\,(\Omega^n)\,, \mu^n)$,*

$$Var\,(f) \leq K \log\left(1/p_{min}\right) \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(e\,\|\Delta_i f\|_2 \,/\, \|\Delta_i f\|_1\right)}. \tag{2.1.3}$$

Remark 2.1.4. Inequality (2.1.3) is sharp up to a universal constant factor, which can easily be seen by taking the function $f(x) = 1$ if $x_i = \omega$ for all $i = 1, \ldots, n$ where ω is some element of Ω is such that $\mu\,(\{\omega\}) = p_{min}$, and $f(x) = 0$ otherwise.

Herein, we follow the line of argument of Talagrand [90] and modify his symmetrization procedure to deduce the result above. Given the paper of Talagrand [90], the proof is self contained apart from Lemma 1 of [35].

Cordero-Erausquin and Ledoux [30] in a recent preprint further generalized Theorem 2.1.3, however their approach is very different from the original proof of Talagrand. (One can deduce a result, equivalent up to a universal constant to Theorem 2.1.3, from Theorem 1 of [30], by combining it with Theorem A.1 of [36]. This results in a slightly more complicated proof.)

We finish this section by noting that the special case of Theorem 2.1.3, where μ^n is the uniform measure on Ω^n, has been proved in [35].

2.1.2 Background and further motivation for Theorem 1.3

Falik and Samarodnitsky [40] used logarithmic Sobolev inequalities to derive edge isoperimetric inequalities. Rossignol used this method to derive sharp threshold results [81, 82]. Furthermore, Benaïm and Rossignol [12] extended the results of [13] (where Talagrand's Theorem 2.1.1 above is applied to first-passage percolation), again with the use of logarithmic Sobolev inequalities. These similar applications suggest a deeper connection between logarithmic Sobolev inequalities and (2.1.2). Indeed, Bobkov and Houdré in [17], proved that a version of (2.1.2) actually implies a logarithmic Sobolev inequality in a continuous set-up. Moreover, Cordero-Erausquin and Ledoux in [30] showed the same implication under different assumptions. See also Section 1.2.5.

Another motivation for Theorem 2.1.3 is to point out the following mistake in the literature. We borrow the notation of [55]. For any $x \in \Omega^n$ and $i = 1, \ldots, n$, we define

$$s_i(x) = \{y \in \Omega^n \mid y_j = x_j \text{ for all } j \neq i\}.$$

For $i = 1, \ldots, n$, let $I_f(i)$ denote the probability of the event that the value of f does depend on the ith coordinate, that is

$$I_f(i) = \mu^n \left(\{x \in \Omega^n \,:\, f \text{ is non-constant on } s_i(x)\}\right).$$

The following claim, which is related to our Theorem 2.1.3 was stated as Theorem 3.3 in [55]. However, as we will show, this claim is incorrect.

For any probability space $(\Omega, \mathcal{F}, \mu)$, and positive integer n, for any square integrable function $f : (\Omega^n, \mathcal{F}^n, \mu^n) \to \mathbb{R}$, we have

$$Var\,(f) \leq 10 \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(1/I_f(i)\right)}. \tag{2.1.4}$$

One can easily see, that the following is a counterexample for this claim. Let k be an arbitrary positive integer. Take $n = 2$ and consider the case where $\Omega = [0, 1]$ and μ is the uniform measure. Take the function f (similar to the function in Remark 2.1.4) defined as $f(x_1, x_2) = 1$ if $0 \leq x_1, x_2 \leq 1/k$ and

0 otherwise. Substituting to (2.1.4) and choosing k large enough, we get a contradiction.

Note that we can easily salvage (2.1.4) under the conditions of Theorem 2.1.3. If in equation (2.1.4) we replace the constant 10 for $K\log(1/p_{min})$, we get a valid statement, since we it follows from (2.1.3) by applying second moment method in the denominator.

Most of the aforementioned applications of the inequality (2.1.2) are concerned with the special case where $f = \mathbf{1}_A$, that is f is the indicator function of some event $A \subseteq \Omega^n$. We warn the reader about the slight inconsistency of the literature: $I_A(i)$ is called the influence of the ith variable on the event A, instead of some $L^p, p \geq 1$ norm of $\Delta_i f = \Delta_i \mathbf{1}_A$, which is the usual influence for arbitrary functions. For comparison of different definitions of influence, see e.g. [64].

Note that

$$\|\Delta_i \mathbf{1}_A\|_2^2 = \|\Delta_i \mathbf{1}_A\|_1 \leq p_{med}\mu^n\left(A_i\right), \tag{2.1.5}$$

where $p_{med} = \max\left\{\mu(B) | B \subset \Omega, \mu(B) \leq \frac{1}{2}\right\}$. Using this we can deduce the following generalization of Corollary 1.2 of [90].

Corollary 2.1.5. *There is a universal constant $C > 0$ such that for each finite set Ω and each measure μ on Ω and for sets $A \subseteq \Omega^n$,*

$$\sum_{i=1}^{n} I_A(i) \geq C\frac{\log\left(1/\max_i I_A(i)\right)}{p_{med}\log\left(1/p_{min}\right)}\mu^n(A)\left(1-\mu^n\left(A\right)\right). \tag{2.1.6}$$

Using the corollary above, one can easily deduce the sharp threshold results of [22].

We finish this introduction with some remarks on the proof of Theorem 2.1.3. The proof of Theorem 1.5 of [90] uses a hypercontractive result (Bonami-Beckner inequality, see [10]) followed by a subtle symmetrization procedure (see Step 2 and 3 of the proof of Lemma 2.1 in [90]). In the proof of our more general Theorem 2.1.3 above, we use a consequence of the extended Bonami-Beckner inequality (for an extension of the Bonami-Beckner inequality see Claim 3.1 in [6]) from [35] and then modify Talagrand's symmetrization procedure. This generalization of Talagrand's symmetrization argument, which covers Sections 2.2.2 and 2.2.3 is the main part of our proof.

Acknowledgement.

The author thanks Rob van den Berg, for introducing him to the subject, and for the comments on drafts of this paper.

2.2 Proof of Theorem 2.1.3

Without loss of generality, we assume that $\Omega = \mathbb{Z}_k$ (the integers modulo k) for some $k \in \mathbb{N}$.

Let η be an arbitrary measure on $\mathbb{Z}_k^n$. For each η, we will write $L_\eta\left(\mathbb{Z}_k^n\right)$ for the (Hilbert) space of complex valued functions on $\mathbb{Z}_k^n$, with the inner product

$$\langle f,g\rangle_\eta = \int_{\mathbb{Z}_k^n} f\bar{g}\,d\eta \text{ for } f,g\in L_\eta\left(\mathbb{Z}_k^n\right).$$

We will write $\|f\|_{L^q(\eta)}$ for the q-norm, $q\in[1,\infty)$, of a function $f:\mathbb{Z}_k^n\to\mathbb{C}$ with respect to the measure η, that is

$$\|f\|_{L^q(\eta)} = \left(\int |f|^q\,d\eta\right)^{1/q}.$$

When it is clear from the context which measure we are working with, we will simply write $\|f\|_q$.

2.2.1 A hypercontractive inequality

Let ν^n denote the uniform measure on $\mathbb{Z}_k^n$. Define the "scalar product" on $\mathbb{Z}_k^n$ by

$$\langle x,y\rangle = \sum_{i=1}^n x_i y_i, \text{ for } x,y\in\mathbb{Z}_k^n.$$

Let $\varepsilon = e^{2\pi i/k}$. For every $y\in\mathbb{Z}_k^n$, define the functions

$$w_y(x) = \varepsilon^{\langle x,y\rangle} \text{ for } x\in\mathbb{Z}_k^n.$$

It is easy to check the following lemma.

Lemma 2.2.1. *$\{w_y\}_{y\in\mathbb{Z}_k^n}$ form an orthonormal basis in $L_{\nu^n}\left(\mathbb{Z}_k^n\right)$.*

Let us denote the number of non-zero coordinates of $\xi\in\mathbb{Z}_k^n$ by $[\xi]$. We will use the following hypercontractive inequality:

Lemma 2.2.2. *(Lemma 1 of [35]) There are positive constants C,γ such such that for any $k,n\in\mathbb{N}$, $m\in\{0,1,\dots,n\}$ and complex numbers a_y, for $y\in\mathbb{Z}_k^n$, we have*

$$\left\|\sum_{[y]=m} a_y w_y\right\|_{L^4(\nu^n)} \le \left(Ck^\gamma\right)^m \left(\sum_{[y]=m}\left|a_y\right|^2\right)^{1/2}. \tag{2.2.1}$$

Remark 2.2.3. The proof (in [35]) of Lemma 2.2.2 is based on Claim 3.1 of [6]. Claim 3.1 of [6] is a generalization of the so called Bonami-Beckner inequality (see Lemma 1 of [10]). That inequality played an important role in [90] in the original proof of Theorem 2.1.1.

2.2.2 Finding a suitable basis

We assume that $\mu(\{j\}) > 0$ for all $j \in \mathbb{Z}_k$. Let $L_\mu(\mathbb{Z}_k)$ be the Hilbert space of functions from $\mathbb{Z}_k$ to $\mathbb{C}$, with the inner product

$$\langle a, b\rangle_\mu = \sum_{j\in\mathbb{Z}_k} a(j)\overline{b(j)}\mu(\{j\}) \text{ for } a, b \in L_\mu(\mathbb{Z}_k).$$

Let $c_0 \in L_\mu(\mathbb{Z}_k)$ be the constant 1 function. By Gram-Schmidt orthogonalization, there exist functions $c_l \in L_\mu(\mathbb{Z}_k)$ for $l \in \mathbb{Z}_k \setminus \{0\}$, such that c_j, $j \in \mathbb{Z}_k$ form an orthonormal basis in $L_\mu(\mathbb{Z}_k)$.

Using the functions c_j, $j \in \mathbb{Z}_k$ we define an orthonormal basis in $L_{\mu^n}(\mathbb{Z}_k^n)$ analogous to the basis w_y, $y \in \mathbb{Z}_k^n$. It is easy to check the following lemma.

Lemma 2.2.4. *The functions u_y, for $y \in \mathbb{Z}_k^n$, defined by*

$$u_y(x) = \prod_{i=1}^{n} c_{y_i}(x_i) \text{ for } x \in \mathbb{Z}_k^n, \tag{2.2.2}$$

form an orthonormal basis in $L_\mu(\mathbb{Z}_k^n)$.

2.2.3 Extension of Lemma 2.2.2

The key ingredient in the proof of Theorem 2.1.3 is the following generalization of Lemma 2.2.2. It can also be seen as an extension of Lemma 2.1 of [90]. One could also use Theorem 2.2 of [99], however the proof of that theorem is more complicated.

Lemma 2.2.5. *With the constants of Lemma 2.2.2, we have for every $k, n \in \mathbb{N}$, $m \in \{0, 1, \ldots, n\}$ and complex numbers a_y, $y \in \mathbb{Z}_k^n$,*

$$\left\| \sum_{[y]=m} a_y u_y \right\|_{L^4(\mu^n)} \le (C\theta k^\gamma)^m \left(\sum_{[y]=m} |a_y|^2 \right)^{1/2} \tag{2.2.3}$$

holds, where $\theta = k \max_{i,j} |c_i(j)|$.

Proof. The proof generalizes the symmetrization technique of the proof of Lemma 2.1 of [90]. Recall the definitions of ε and w_y for $y \in \mathbb{Z}_k^n$ of Section 2.2.1. Let n, k, m and the numbers a_y $y \in \mathbb{Z}_k^n$ as in the statement of Lemma 2.2.2.

Step 1 Define the product space $G = (\mathbb{Z}_k^n)^k$ with the probability measure $\mu_k^n = \bigotimes_{i=1}^k \mu$. For $y, z \in \mathbb{Z}_k^n$ define the functions $g_y, g_{y,z}$ on G as follows. For $X = (X^0, \ldots, X^{k-1}) \in (\mathbb{Z}_k^n)^k$ and $z \in \mathbb{Z}_k^n$, let

$$g_y(X) = \prod_{1\le i\le n,\, y_i\ne 0} \sum_{l=0}^{k-1} c_{y_i}(X_i^l)\varepsilon^{l y_i}, \tag{2.2.4}$$

$$g_{y,z}\left(X\right)=\prod_{1\le i\le n,\,y_i\neq 0}\varepsilon^{z_iy_i}\sum_{l=0}^{k-1}c_{y_i}(X_i^l)\varepsilon^{ly_i}=g_y\left(X\right)w_y(z). \tag{2.2.5}$$

Recall that ν is the uniform measure on $\mathbb{Z}_k^n$, and define the set $H=G\times\mathbb{Z}_k^n$ and the product measure $\kappa=\mu_k\otimes\nu$ on H. We also define, for $y\in\mathbb{Z}_k^n$ the functions h_y on H by $h_y\left(X,z\right)=g_{y,z}\left(X\right)=g_y\left(X\right)w_y(z)$.

Step 2 For X as before and for $z\in\mathbb{Z}_k^n$ define X_z as

$$\left(X_z\right)_i^l=X_i^{l+z_i \bmod k}.$$

Then

$$\begin{aligned} g_{y,z}\left(X_z\right)&=\prod_{1\le i\le n,\,y_i\neq 0}\sum_{l=0}^{k-1}c_{y_i}(X_i^{l+z_i \bmod k})\varepsilon^{(l+z_i)y_i}\\ &=\prod_{1\le i\le n,\,y_i\neq 0}\sum_{l=0}^{k-1}c_{y_i}(X_i^l)\varepsilon^{ly_i}=g_y\left(X\right). \end{aligned}$$

Hence for each fixed $z\in\mathbb{Z}_k^n$, we have

$$\left\|\sum_{[y]=m}a_yg_y\right\|_{L^4\left(\mu_k^n\right)}=\left\|\sum_{[y]=m}a_yg_{y,z}\right\|_{L^4\left(\mu_k^n\right)}. \tag{2.2.6}$$

Integrating over the variable z with respect to ν^n, Fubini's theorem gives that

$$\left\|\sum_{[y]=m}a_yg_y\right\|_{L^4\left(\mu_k^n\right)}=\left\|\sum_{[y]=m}a_yh_y\right\|_{L^4(\kappa)}. \tag{2.2.7}$$

Step 3 For fixed X, use Lemma 2.2.2 for the numbers $a_yg_y\left(X\right)$, and get

$$\int\left|\sum_{[y]=m}a_yg_y\left(X\right)w_y\left(z\right)\right|^4d\nu^n(z)\le\left(Ck^\gamma\right)^{4m}\left(\sum_{[y]=m}\left|a_yg_y\left(X\right)\right|^2\right)^2. \tag{2.2.8}$$

Since $\theta=k\max_{i,j}\left|c_i\left(j\right)\right|$, we have that $\left|g_y\left(X\right)\right|\le\theta^m$, which together with (2.2.8) gives

$$\int\left|\sum_{[y]=m}a_yg_y\left(X\right)w_y\left(z\right)\right|^4d\nu^n(z)\le\left(C\theta k^\gamma\right)^{4m}\left(\sum_{[y]=m}\left|a_y\right|^2\right)^2.$$

Integrating with respect to $d\mu_k(X)$ and taking the 4th root gives

$$\left\|\sum_{[y]=m}a_yh_y\right\|_{L^4(\kappa)}\le\left(C\theta k^\gamma\right)^m\left(\sum_{[y]=m}\left|a_y\right|^2\right)^{1/2}. \tag{2.2.9}$$

By (2.2.9) and (2.2.7) we only have to show that

$$\left\| \sum_{[y]=m} a_y u_y \right\|_{L^4(\mu^n)} \le \left\| \sum_{[y]=m} a_y g_y \right\|_{L^4\left(\mu_k^n\right)}. \tag{2.2.10}$$

Step 4 Now we prove an alternative form of the function g_y. Recall the definition (2.2.4) of g_y. Expand the product, and get

$$\begin{aligned} g_y(X) &= \prod_{1\le i\le n,\, y_i\ne 0} \sum_{l=0}^{k-1} c_{y_i}(X_i^l)\varepsilon^{ly_i} \\ &= \sum_{\alpha:(*)} \prod_{1\le i\le n,\, y_i\ne 0} c_{y_i}(X_i^{\alpha(i)})\varepsilon^{\alpha(i)y_i}, \end{aligned} \tag{2.2.11}$$

where $(*)$ denotes the sum over all functions $\alpha : \{i \,|\, y_i \neq 0\} \to \mathbb{Z}_k$.

We will use the following trivial observation:

Observation: $c_{y_i}(X_i^l)\varepsilon^{ly_i} = 1$ whenever $y_i = 0$.

With the Observation we rewrite (2.2.11) as follows.

$$\begin{aligned} g_y(X) &= \sum_{\alpha\in\mathcal{A}_y} \prod_{i=1}^{n} c_{y_i}(X_i^{\alpha(i)})\varepsilon^{\alpha(i)y_i} \\ &= \sum_{\alpha\in\mathcal{A}_y} \prod_{t\in\mathbb{Z}_k} \prod_{1\le i\le n,\, \alpha(i)=t} c_{y_i}(X_i^t)\varepsilon^{ty_i}, \end{aligned} \tag{2.2.12}$$

where $\mathcal{A}_y$ is the set of functions $\alpha : \{1, 2, \ldots, n\} \to \mathbb{Z}_k$ with the property that $\alpha(i) = 0$ if $y_i = 0$. For a function $\alpha \in \mathcal{A}_y$ we can define the vectors $v^t = v^t(\alpha) \in \mathbb{Z}_k^n$ for $t \in \mathbb{Z}_k$ by

$$v_i^t = v_i^t(\alpha) = \begin{cases} y_i & \text{if } \alpha(i) = t \\ 0 & \text{otherwise.} \end{cases}$$

The map $\alpha \mapsto (v^t(\alpha))_{t\in\mathbb{Z}_k}$ is one-to-one, furthermore the image of $\mathcal{A}_y$ under this map is

$$\mathcal{V}_y = \left\{ v = \left(v^t\right)_{t\in\mathbb{Z}_k} \,\middle|\, \sum_{t\in\mathbb{Z}_k} v^t = y, \text{ and } \forall i\, v_i^t \neq 0 \text{ for at most one } t \in \mathbb{Z}_k \right\}.$$

Using the properties of the map $\alpha \mapsto (v^t(\alpha))_{t\in\mathbb{Z}_k}$ together with the Observation and the definition of u, we can conclude from (2.2.12) that

$$g_y(X) = \sum_{v\in\mathcal{V}_y} \prod_{t\in\mathbb{Z}_k} \prod_{i=1}^{n} c_{v_i^t}(X_i^t)\varepsilon^{tv_i^t}$$

$$= \sum_{v\in\mathcal{V}_y}\prod_{t\in\mathbb{Z}_k} u_{v^t}(X^t)\varepsilon^{t\langle v^t,\mathbf{1}\rangle} \tag{2.2.13}$$

where $\mathbf{1}$ is vector in $\mathbb{Z}_k^n$ with all coordinates equal to 1.

Step 5 Now we prove (2.2.10). Jensen's inequality gives that

$$\begin{aligned}
&\int\left|\sum_{[y]=m} a_y g_y(X)\right|^4 d\mu_k^n(X)\\
&\qquad\ge \int\left|\int\sum_{[y]=m} a_y g_y(X)\, d\mu_{k-1}^n\left(X^1,\dots,X^{k-1}\right)\right|^4 d\mu^n\left(X^0\right)\\
&\qquad= \int\left|\sum_{[y]=m} a_y\int g_y(X)\, d\mu_{k-1}^n\left(X^1,\dots,X^{k-1}\right)\right|^4 d\mu^n\left(X^0\right). \qquad (2.2.14)
\end{aligned}$$

By (2.2.13), the inner integral of the left hand side of (2.2.14) is

$$\begin{aligned}
&\int g_y(X) d\mu_{k-1}^n\left(X^1,\dots,X^{k-1}\right)\\
&\qquad= \int\sum_{v\in\mathcal{V}_y}\prod_{t\in\mathbb{Z}_k} u_{v^t}(X^t)\varepsilon^{t\langle v^t,\mathbf{1}\rangle} d\mu_{k-1}^n\left(X^1,\dots,X^{k-1}\right) \qquad (2.2.15)\\
&\qquad= \sum_{v\in\mathcal{V}_y}\left(\prod_{t\in\mathbb{Z}_k}\varepsilon^{t\langle v^t,\mathbf{1}\rangle}\right) u_{v^0}(X^0)\prod_{l=1}^{k-1}\int u_{v^l}(X^l) d\mu^n\left(X^l\right). \qquad (2.2.16)
\end{aligned}$$

Since u_0 is the constant 1 function on $\mathbb{Z}_k^n$, and by Lemma 2.2.4 $(u_w,\ w\in\mathbb{Z}_k^n)$ is an orthonormal basis of $L_\mu\left(\mathbb{Z}_k^n\right)$, we have

$$\int u_w d\mu^n = \int u_w u_0 d\mu^n = \begin{cases} 1 & \text{if } w=0\\ 0 & \text{otherwise.}\end{cases}$$

By this and the definition of $\mathcal{V}_y$ we conclude from (2.2.16) that

$$\begin{aligned}
&\int g_y(X) d\mu_{k-1}^n\left(X^1,\dots,X^{k-1}\right)\\
&\qquad= \sum_{v\in\mathcal{V}_y,\, v^1=\ldots=v^{k-1}=0}\left(\prod_{t\in\mathbb{Z}_k}\varepsilon^{t\langle v^t,\mathbf{1}\rangle}\right) u_{v^0}(X^0) = u_y(X^0). \qquad (2.2.17)
\end{aligned}$$

(2.2.17) together with (2.2.14) gives that

$$\int\left|\sum_{[y]=m} a_y g_y(X)\right|^4 d\mu_k^n(X) \ge \int\left|\sum_{[y]=m} a_y u_y(X^0)\right|^4 d\mu^n\left(X^0\right),$$

from which by taking the 4th root, we get (2.2.10). This completes the proof of Lemma (2.2.5). □

From Lemma (2.2.5) and duality, we conclude the following lemma.

Lemma 2.2.6. *With the constants of Lemma 2.2.2, for any function $g \in L_\mu(\mathbb{Z}_k^n)$ we have*

$$\sum_{[y]=l} |\hat{g}(y)|^2 \leq (C\theta k^\gamma)^{2l} \|g\|^2_{L^{4/3}(\mu)}.$$

2.2.4 Completion of the proof of Theorem 2.1.3

Notice that

$$\begin{aligned}\int_\Omega u_y(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_n)\,\mu(d\xi) &= \sum_{j\in\mathbb{Z}_k} c_{y_i}(j)\mu(\{j\}) \prod_{1\leq l\leq n\, l\neq i} c_{y_l}(x_l)\\ &= \langle c_{y_i}, c_0\rangle_\mu \prod_{1\leq l\leq n\, l\neq i} c_{y_l}(x_l)\\ &= \begin{cases} u_y(x) & \text{if } y_i = 0\\ 0 & \text{if } y_i \neq 0.\end{cases}\end{aligned}$$

Hence

$$\int_\Omega f(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_n)\,\mu(d\xi) = \sum_{y\in\mathbb{Z}_k^n,\, y_i=0} \hat{f}(y)u_y$$

where $f = \sum_y \hat{f}(y)u_y$, i.e $\hat{f}(y) = \langle f, u_y\rangle_\mu$.
By the definition of $\Delta_i f$ we have

$$\Delta_i f = \sum_{y\in\mathbb{Z}_k^n,\, y_i\neq 0} \hat{f}(y)u_y. \tag{2.2.18}$$

Recall that $[y]$ was the number of non-zero coordinates of a vector $y \in \mathbb{Z}_k$. Define $M(g)$ by

$$M(g)^2 = \sum_{y\in\mathbb{Z}_k^n,\, y\neq 0} \frac{\hat{g}(y)^2}{[y]} \text{ for } g \in L_\mu(\mathbb{Z}_k^n).$$

Take a function $f \in L_\mu(\mathbb{Z}_k^n)$ with $\int f d\mu = 0$ (which is equivalent to $\hat{f}(0) = 0$). Then Parseval's formula and (2.2.18) gives that

$$\|f\|^2_{L^2(\mu^n)} = \sum_{y\neq 0} \hat{f}(y)^2 = \sum_{i=1}^n M(\Delta_i f)^2. \tag{2.2.19}$$

Since $1 = \sum_{j=0}^{k-1} |c_i(j)|^2 p_j$, we can conclude that $\theta \leq k/\min_j \sqrt{p_j}$. Hence Theorem 2.1.3 follows from the Proposition 2.2.7 below and (2.2.19).

Proposition 2.2.7. *There is a positive constant K, such that if $\int g d\mu = 0$, we have*

$$M(g)^2 \leq K \log\left(C\theta k^{\gamma}\right) \frac{\|g\|_2^2}{\log\left(e\,\|g\|_2 \,/\, \|g\|_1\right)},$$

where $\theta = k \max_{i=1,\ldots,n\, j\in\mathbb{Z}_k} |c_i\,(j)|$, and the constants C, γ are the same as in Lemma 2.2.2.

Proof. The proof of Proposition (2.2.7) is the same as the proof of Proposition 2.3 in [90] with the following modifications. Take $q = 4$ instead of $q = 3$, and use Lemma 2.2.6 instead of Proposition 2.2 of [90]. The only difference will be in the constants. First we get the term $2\log\left(C\theta k^{\gamma}\right)$ in stead of $\log\left(2\theta^2\right)$. Furthermore we have to replace the estimate

$$\frac{\|g\|_2}{\|g\|_1} \leq \left(\frac{\|g\|_2}{\|g\|_{3/2}}\right)^3$$

by

$$\frac{\|g\|_2}{\|g\|_1} \leq \left(\frac{\|g\|_2}{\|g\|_{4/3}}\right)^2,$$

which is a consequence of the Cauchy-Schwartz inequality. This substitution only affects the constant K.

This completes the proof of Proposition (2.2.7) and the proof of Theorem 2.1.3. □

3 Sublinearity of the travel-time variance for dependent first passage percolation

This chapter is based on the paper [95] with Jacob van den Berg.

Abstract

Let E be the set of edges of the d-dimensional cubic lattice $\mathbb{Z}^d$, with $d \geq 2$, and let $t(e), e \in E$, be non-negative values. The passage time from a vertex v to a vertex w is defined as $\inf_{\pi : v \to w} \sum_{e \in \pi} t(e)$, where the infimum is over all paths π from v to w, and the sum is over all edges e of π.

Benjamini, Kalai and Schramm [13] proved that if the $t(e)$'s are i.i.d. two-valued positive random variables, the variance of the passage time from the vertex 0 to a vertex v is sublinear in the distance from 0 to v. This result was extended to a large class of independent, continuously distributed t-variables by Benaïm and Rossignol [12].

We extend the result by Benjamini, Kalai and Schramm in a very different direction, namely to a large class of models where the $t(e)$'s are dependent. This class includes, among other interesting cases, a model studied by Higuchi and Zhang [57], where the passage time corresponds with the minimal number of sign changes in a subcritical 'Ising landscape'.

Keywords and phrases. first-passage percolation, influence results, greedy lattice animals, Ising model.
AMS 2010 classifications. Primary 60K35; Secondary 82B43.

3.1 Introduction and statement of results

Consider, for $d \geq 2$, the d-dimensional lattice $\mathbb{Z}^d$. Let $\mathbb{E}$ denote the set of edges of the lattice, and let $t(e), e \in \mathbb{E}$ be non-negative real values. A path from a vertex v to a vertex w is an alternating sequence of vertices and edges

$$v_0 = v, e_1, v_1, e_2, \cdots, v_{n-1}, e_n, v_n = w,$$

where each e_i is an edge between the vertices v_{i-1} and v_i, $1 \leq i \leq n$. To indicate that e is an edge of a path π, we often write, with some abuse of notation, $e \in \pi$.

If $v = (v_1, \cdots, v_d)$ is vertex, we use the notation $|v|$ for $\sum_{i=1}^{d} |v_i|$. The (graph) distance $d(v, w)$ between vertices v and w is defined as $|v - w|$. The vertex $(0, \cdots, 0)$ will be denoted by 0.

The passage time of a path π is defined as

$$T(\pi) = \sum_{e \in \pi} t(e). \tag{3.1.1}$$

The passage time (or travel time) $T(v, w)$ from a vertex v to a vertex w is defined as

$$T(v, w) = \inf_{\pi : v \to w} T(\pi),$$

where the infimum is over all paths π from v to w.

Analogously to the above described bond version, there is a natural site version of these notions: In the site version the t variables are assigned to the vertices instead of the edges. In the definition of $T(\pi)$ the r.h.s. in (3.1.1) is then replaced by its analogue where the sum is over all vertices of π. There seems to be no 'fundamental' difference between the bond and the site version.

An important subject of study in first-passage percolation is the asymptotic behaviour of $T(0, v)$ and it fluctuations, when $|v|$ is large and the $t(e)$'s are random variables. It is believed that, for a large class of distributions of the $t(e)$'s, the variance of $T(0, v)$ is of order $|v|^{2/3}$. However, this has only been proved for a special case in a modified (oriented) version of the model [61]. Apart from this, the best upper bounds obtained for the variance before 2003 were linear in $|v|$ [65]. See Section 1 of [13] for more background and references.

Benjamini, Kalai and Schramm [13] showed that if the $t(e)$'s are i.i.d. random variables taking values a and b, $b \geq a > 0$, then the variance of $T(0, v)$ is sublinear in the distance from 0 to v. More precisely, they showed the following theorem.

Theorem 3.1.1. [Benjamini, Kalai and Schramm [13]] *Let $b \geq a > 0$. If the $(t(e), e \in \mathbb{E})$ are i.i.d random variables taking values in $\{a, b\}$, then there is a constant $C > 0$ such that, for all v with $|v| \geq 2$,*

$$Var(T(0, v)) \leq C \frac{|v|}{\log |v|}. \tag{3.1.2}$$

Benaïm and Rossignol [12] extended this result to a large class of i.i.d. t-variables with a continuous distribution, and also proved concentration results. See also [48].

We give a generalization of Theorem 3.1.1 in a very different direction, namely to a large class of *dependent* t-variables. The description of this class, and the statement of our general results are given in Subsection 3.1.4.

Using our general results we show in particular that (3.1.2) holds for the $\{a, b\}$-valued Ising model with $0 < a < b$ and inverse temperature $\beta < \beta_c$. By $\{a, b\}$ valued Ising model we mean the model that is simply obtained from the ordinary, $\{-1, +1\}$-valued, Ising model by replacing -1 by a and $+1$ by b. The precise definition of the Ising model and the statement of this result is given in Subsection 3.1.1.

We also study, as a particular case of our general results, a different Ising-like first passage percolation model: Consider an 'ordinary' Ising model (with signs -1 and $+1$), with parameters $\beta < \beta_c$ and with external field h satisfying certain conditions. Now define the passage time $T(v, w)$ between two vertices v and w as the minimum number of sign changes needed to travel from v to w. Higuchi and Zhang [57] proved, for $d = 2$, a concentration result for this model. This concentration result implies an upper bound for the variance that is (a 'logarithmic-like' factor) *larger* than linear. We show from our general framework that the sublinear bound (3.1.2) holds (see Theorem 3.1.5).

The last special case we mention explicitly is that where the collection of t-variables is a finite-valued Markov random field which satisfies a high-noise condition studied by Häggström and Steif (see [52]). Again it follows from our general results that the sublinear bound (3.1.2) holds (see Theorem 3.1.4).

The general organization of the paper is as follows: In the next three subsections we give precise definitions and statements concerning the special results mentioned above. Then, in Subsection 3.1.4, we state our main, more general results, Theorems 3.1.6 and 3.1.7.

In Section 3.2 we prove the special cases (Theorems 3.1.2, 3.1.4 and 3.1.5) from Theorem 3.1.6 and Theorem 3.1.7.

In Section 3.3 we present the main ingredients for the proofs of our general results: an inequality by Talagrand (and its extension to multiple-valued random variables), a very general 'randomization tool' of Benjamini, Kalai and Schramm, and a result on greedy lattice animals by Martin [75].

In Section 3.4 we first give a very brief informal sketch of the proof of Theorem 3.1.6 (pointing out the extra problems that arise, compared with the i.i.d. case in [13]), followed by a formal, detailed proof.

The proof of Theorem 3.1.7 is very similar to that of Theorem 3.1.6. This is explained in Section 3.5.

3.1.1 The case where the t-variables have an $\{a, b\}$ valued Ising distribution

Recall that the Ising model (with inverse temperature β and external field h) on a countably infinite, locally finite graph G is defined as follows. First some notation: We write $v \sim w$ to indicate that two vertices v and w share an edge. For each vertex v of G, the set of vertices $\{v : w \sim v\}$ is denoted by ∂v. The spin value ($+1$ or -1) at a vertex v is denoted by σ_v. Now define, for each vertex v and each $\alpha \in \{-1, +1\}^{\partial v}$, the distribution $q_v^\alpha = q_{v;\beta,h}^\alpha$, on $\{-1, +1\}$:

$$q_v^\alpha(+1) = \frac{\exp\left(\beta(h + \sum_{w\sim v} \alpha_w)\right)}{\exp\left(\beta(h + \sum_{w\sim v} \alpha_w)\right) + \exp\left(-\beta(h + \sum_{w\sim v} \alpha_w)\right)}, \qquad (3.1.3)$$
$$q_v^\alpha(-1) = \frac{\exp\left(-\beta(h + \sum_{w\sim v} \alpha_w)\right)}{\exp\left(\beta(h + \sum_{w\sim v} \alpha_w)\right) + \exp\left(-\beta(h + \sum_{w\sim v} \alpha_w)\right)}.$$

Let V denote the set of vertices of G. An Ising distribution on G (with parameters β and h) is a probability distribution $\mu_{\beta,h}$ on $\{-1, +1\}^V$ which satisfies, for each vertex v and each $\eta \in \{-1, +1\}$,

$$\mu_{\beta,h}(\sigma_v = \eta \,|\, \sigma_w, w \neq v) = q_v^{\sigma_{\partial v}}(\eta), \;\; \mu_{\beta,h} - \text{a.s.} \qquad (3.1.4)$$

In this (usual) set-up, the spin values are assigned to the vertices. One can define an Ising model with spins assigned to the edges, by replacing G by its cover graph. (That is, the graph whose vertices correspond with the edges of G, and where two vertices share an edge if the edges of G to which these vertices correspond, have a common endpoint).

In the case where G is the $d-$ dimensional cubic lattice $\mathbb{Z}^d$, with $d \geq 2$, it is well-known that there is a critical value $\beta_c \in (0, \infty)$ such that the following holds: If $\beta < \beta_c$, there is a unique distribution satisfying (3.1.4). If $\beta > \beta_c$ and $h = 0$ there is more than one distribution satisfying (3.1.4). A similar result (but with a different value of β_c) holds for the edge version of the model.

Let $b > a > 0$. An $\{a, b\}$ *valued Ising model* is obtained from the usual Ising model by reading a for -1 and b for $+1$. More precisely, if $(\sigma_v, v \in V)$ has an Ising distribution and, for each $v \in V$, $t(v)$ is defined to be a if $\sigma_v = -1$ and b if $\sigma_v = +1$, then we say that $(t(v), v \in V)$ are $\{a, b\}$-valued Ising variables. A similar definition holds for the situation where the spins are assigned to the edges.

A special case of our main result is the following extension of Theorem 3.1.1 to the Ising model.

Theorem 3.1.2. *Let $b > a > 0$ and $d \geq 2$. If $(t(v), v \in \mathbb{Z}^d)$ are $\{a, b\}$-valued Ising variables with inverse temperature $\beta < \beta_c$ and external field h, then there is a constant $C > 0$ such that for all v with $|v| \geq 2$,*

$$Var(T(0, v)) \leq C \frac{|v|}{\log |v|}. \qquad (3.1.5)$$

The analogue of this result holds for the case where the values a, b are assigned to the edges.

3.1.2 Markov random fields with high-noise condition

Let $(\sigma_v, v \in \mathbb{Z}^d)$, be a translation invariant Markov random field taking values in $W^{\mathbb{Z}^d}$ where W is a finite set. Let $v \in \mathbb{Z}^d$. For each $w \in W$ define (see [52]),

$$\gamma_w = \min_{\eta \in W^{\partial v}} \mathbb{P}\left(\sigma_v = w | \sigma_{\partial v} = \eta\right).$$

Further, define

$$\gamma = \sum_{w \in W} \gamma_w.$$

Note that the definition of γ_w and γ does not depend on the choice of v. Häggström and Steif [52] studied the existence of finitary codings (and exact simulations) of Markov random fields under the following high noise (HN) condition (see also [51] and [92]):

Definition 3.1.3. *[HN condition] A translation invariant Markov random field on $\mathbb{Z}^d$ satisfies the HN condition, if*

$$\gamma > \frac{2d-1}{2d}.$$

We will show that the following theorem is a consequence of our main result.

Theorem 3.1.4. *Let $d \geq 2$ and let $(\sigma_v, v \in \mathbb{Z}^d)$ be a translation invariant Markov random field taking finitely many, strictly positive values. If this Markov random field satisfies the HN condition, then, for the first-passage percolation model with $t(v) = \sigma_v, v \in \mathbb{Z}^d$, there is a constant $C > 0$ such that for all v with $|v| \geq 2$,*

$$Var(T(0, v)) \leq C \frac{|v|}{\log |v|}. \tag{3.1.6}$$

The analogue of this result holds for the edge version of the model.

Remark. The HN condition for the edge version is a natural modification of that in Definition 3.1.3. For instance, the $2d$ in the numerator and the denominator of the r.h.s. of the inequality in Definition 3.1.3 is the number of nearest-neighbour vertices of a given vertex, and will be replaced by $4d - 2$ (which is the number of edges sharing an endpoint with a given edge).

3.1.3 The minimal number of sign changes in an Ising pattern

In Subsection 3.1.1 the collection of random variables $(t(v), v \in \mathbb{Z}^d)$ itself had an Ising distribution (with -1 and $+1$ translated to a, respectively b). A quite different first-passage percolation process related to the Ising model is the one, studied by Higuchi and Zhang [57], where one counts the minimal number of sign changes from a vertex v to a vertex w in an Ising configuration.

For $\beta < \beta_c$, let $\theta(\beta, h)$ denote the probability that 0 belongs to an infinite $+$ cluster, and let

$$h_c(\beta) = \sup\{h : \theta(\beta, h) = 0\}.$$

For $d = 2$ it was proved in [56] that $h_c(\beta) > 0$.

Using our general results we will prove (in Section 3.2) the following extension of Theorem 3.1.1.

Theorem 3.1.5. *Let the collection of random variables $(\sigma_v,\, v \in \mathbb{Z}^2)$ have an Ising distribution with parameters $\beta < \beta_c$ and external field h, with $|h| < h_c$. Define, for each edge $e = (v_1, v_2)$,*

$$t(e) = \begin{cases} 1 & \text{if } \sigma_{v_1} \neq \sigma_{v_2}, \\ 0 & \text{if } \sigma_{v_1} = \sigma_{v_2}. \end{cases}$$

For the first-passage percolation model with these t-values, there is a $C > 0$ such that for all v with $|v| \geq 2$,

$$\operatorname{Var}(T(0, v)) \leq C \frac{|v|}{\log |v|}. \tag{3.1.7}$$

Remark. The paper [57] by Higuchi and Zhang gives a concentration result for this model (see Theorem 2 in [57]). Their method is very different from ours. (It is interesting to note that the paragraph below (1.11) in their paper suggests that Talagrand-like inequalities are not applicable to the Ising model). The upper bound for the variance of $T(0, v)$ which follows from their concentration result is (a 'logarithmic-like' factor) *larger* than linear. For earlier results on this and related models, see the Introduction in [57].

3.1.4 Statement of the main results

Our main results, Theorem 3.1.6 and Theorem 3.1.7, involve t-variables that can be represented by (or 'encoded' in terms of) i.i.d. finite-valued random variables in a suitable way, satisfying certain conditions. These conditions are of the same flavour as (but somewhat different from) those in Section 2 in [93].

We first need some notation and terminology. Let S be a finite set, and I a countably infinite set. Let W be a finite subset of I. If $x \in S^I$, we write x_W to denote the tuple $(x_i, i \in W)$. If $h : S^I \to \mathbb{R}$ is a function, and $y \in S^W$, we say that y determines the value of h if $h(x) = h(x')$ for all x, x' satisfying $x_W = x'_W = y$.

Let $X_i, i \in I$, be i.i.d. S valued random variables. We say that the random variables $t(v), v \in \mathbb{Z}^d$, are represented by the collection $(X_i, i \in I)$, if, for each $v \in \mathbb{Z}^d$, $t(v)$ is a function of $(X_i, i \in I)$. The formulation of our main theorems involve certain conditions on such a representation:

- *Condition (i)*: There exist $c_0 > 0$ and $\varepsilon_0 > 0$ such that for each $v \in \mathbb{Z}^d$ there is a sequence $i_1(v), i_2(v), \cdots$ of elements of I, such that for all $k = 1, 2, \cdots$,
$$P\left((X_{i_1(v)}, \cdots, X_{i_k(v)}) \text{ does not determine } t(v)\right) \leq \frac{c_0}{k^{3d+\varepsilon_0}}. \tag{3.1.8}$$
- *Condition (ii)*:
$$\begin{gathered}\exists \alpha > 0\ \forall v, w \in \mathbb{Z}^d\ \forall k < \alpha|v-w|, \\ \{i_1(v), \cdots, i_k(v)\} \cap \{i_1(w), \cdots, i_k(w)\} = \emptyset.\end{gathered} \tag{3.1.9}$$
- *Condition (iii)*: The distribution of the family of random variables $(t(v), v \in \mathbb{Z}^d)$ is translation-invariant.

We say that the family of random variables $\left(t\left(v\right), v \in \mathbb{Z}^d\right)$ has a representation satisfying Conditions (i)-(iii), if there are S, I and i.i.d. S-valued random variables $X_i, i \in I$ as above, such that the t-variables are functions of the X-variables satisfying Conditions (i)-(iii) above.

Analogues of these definitions for t-variables indexed by the edges of $\mathbb{Z}^d$ can be given in a straightforward way.

Now we are ready to state our main theorem.

Theorem 3.1.6. *Let $b > a > 0$, and let, with $d \geq 2$, $\left(t\left(v\right), v \in \mathbb{Z}^d\right)$ be a family of random variables that take values in the interval $[a, b]$ and have a representation satisfying Conditions (i)-(iii) above. Then there is a $C > 0$, such that for all $v \in \mathbb{Z}^d$ with $|v| \geq 2$,*
$$\mathrm{Var}(T(0, v)) \leq \frac{C|v|}{\log|v|}. \tag{3.1.10}$$
The analogue for the bond version of this result also holds.

If the t variables can take values equal or arbitrarily close to 0, we need a stronger version of Condition (i) and extra Condition (iv) (see below).
By an optimal path from v to w we mean a path π from v to w such that $T(\pi) \leq T(\pi')$ for all paths π' from v to w.

- *Condition (i')*
There exist $c_0 > 0$, $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$, such that for each $v \in \mathbb{Z}^d$ there is a sequence $i_1(v), i_2(v), \cdots$ of elements of I, such that for all $k = 1, 2, \cdots$,
$$P\left((X_{i_1(v)}, \cdots, X_{i_k(v)}) \text{ does not determine } t(v)\right) \leq c_0 \exp(-\varepsilon_0 k^{\varepsilon_1}). \tag{3.1.11}$$

- *Condition (iv).*
  There exist $c_1, c_2, c_3 > 0$ such that for all vertices v, w the probability that there is no optimal path π from v to w with $|\pi| \le c_1|v - w|$ is at most $c_2 \exp(-c_3|v - w|)$.

Theorem 3.1.7. *Let $b > 0$, and let, with $d \ge 2$, $\left(t\left(v\right), v \in \mathbb{Z}^d\right)$ be a collection of random variables taking values in the interval $[0, b]$, and having a representation satisfying Conditions (i'), (ii), (iii) and (iv) above. Then there is a $C > 0$, such that for all $v \in \mathbb{Z}^d$ with $|v| \ge 2$,*

$$\mathrm{Var}(T(0, v) \le \frac{C|v|}{\log|v|}. \tag{3.1.12}$$

The analogue of this result for the bond version of the model also holds.

Remarks. (a) Note that condition (iii) is in terms of the t-variables only: We do *not* assume that the index set I has a 'geometric' structure and that the t-variables are 'computed' from the X-variables in a 'translation-invariant' way with respect to that structure (and the structure of $\mathbb{Z}^d$).
(b) The goal of our paper is to show that the main result in [13], although its proof heavily uses inequalities concerning independent random variables, can be extended to an interesting class of dependent first-passage percolation models. In the set-up of the above conditions (i), (ii), (iii), (i') and (iv), we have aimed to obtain fairly general Theorems 3.1.6 and 3.1.7, without becoming too general (which would give rise to so many extra technicalities that the main line of argument would be obscured). For instance, from the proofs it will be clear that there is a kind of 'trade-off' between conditions (i) and (ii): one may simultaneously strengthen the first and weaken the second condition.
Also, if the bound in Condition (i') is replaced by a polynomial bound with sufficiently high degree, Theorem 3.1.7 would still hold (but more explanation would be needed in Section 3.5). Since the main motivation for adding this theorem to Theorem 3.1.6 is to handle the interesting Ising sign-change model studied by Higuchi and Zhang (for which we know that Condition (i') holds) we have not replaced Condition (i') by a weaker condition.

Acknowledgements

Our interest in this subject was triggered by a lecture series on first-passage percolation by Vladas Sidoravicius in the fall of 2009.
We thank Chuck Newman for referring us to certain results on greedy lattice animals.

3.2 Proof of Theorems 3.1.2, 3.1.4 and 3.1.5 from Theorems 3.1.6 and 3.1.7

3.2.1 Proof of Theorem 3.1.2

In [93] the notion 'nice finitary representation' has been introduced in the context of 2-dimensional random fields. See conditions (i) - (iv) in Section 2 of that paper. In Section 2 (see in particular Theorem 2.3) in that paper it is shown that the Ising model with $\beta < \beta_c$ has such a representation. (See also [92]). The key ideas and ingredients are exact simulation by coupling from the past (see [79] and [92]), and a well-known result by Martinelli and Olivieri [76] that under a natural dynamics (single-site updates; Gibbs sampler) the system has exponential convergence to the Ising distribution. The random variables used to execute these updates are taken as the X variables in the definition of a representation.

Condition (ii) in [93] is somewhat weaker than our current Condition (i). However, as shown in [93] (see the arguments between Theorem 2.3 and 2.4 in [93]), the above mentioned exponential convergence shows that the Ising model satisfies an even stronger bound, namely Condition (i') in our paper.

Condition (iii) in [93] corresponds with our condition (ii), and Condition (iv) in [93] is stronger than our Condition (iii).

In [93] only the two-dimensional case is treated (because the applications are to percolation models where typical two-dimensional methods are used) but its arguments concerning 'nice finitary representations' for the Ising model extend immediately to higher dimensions.

From the above considerations it follows that the Ising models in the statement of our Theorem 3.1.2 indeed have a representation satisfying our Conditions (i)-(iii). Application of Theorem 3.1.6 now gives Theorem 3.1.2.

3.2.2 Proof of Theorem 3.1.4

The argument is very similar to that in the proof of Theorem 3.1.2. Therefore we only mention the points that need extra attention.

As in the proof of Theorem 3.1.2, the role of the X variables in Section 3.1.4 is played by the i.i.d. random variables driving a single-site update scheme (Gibbs sampler). In Theorem 3.1.2 a form of exponential convergence for the Gibbs sampler was used. This exponential convergence came from a result in [76]. In the current situation the exponential convergence is, as shown in Proposition 2.1 in [52], a consequence of the HN condition. This exponential convergence implies (again, as in the case of Theorem 3.1.2) Condition (i) (and, in fact, the stronger Condition (i')) in Section 3.1.4. Condition (iii) is obvious, and Condition (ii) follows easily (as in the proof of Theorem 3.1.2) from the general set-up of the Gibbs sampler. So, again, we now apply Theorem 3.1.6 to obtain Theorem 3.1.4.

3.2.3 Proof of Theorem 3.1.5

Since $\beta < \beta_c$, the collection $(\sigma_v, v \in \mathbb{Z}^2)$, has (as pointed out in the proof of Theorem 3.1.2) a representation satisfying conditions (i), (ii) and (iii). In fact, as noted in the proof of Theorem 3.1.2, it even satisfies the stronger form (i') of Condition (i). Since $(t(e)$ is a function of the σ-values of the two endpoints of e, it follows immediately that the collection $(t(e), e \in E)$ (where E denotes the set of edges of the lattice $\mathbb{Z}^2$) satisfies the (bond analogue of) the conditions (i'), (ii) and (iii). The fact that (iv) is satisfied follows immediately from Lemma 6 (and (1.9)) in [57]. Theorem 3.1.5 now follows from (the bond version of) Theorem 3.1.7.

3.3 Ingredients for the proof of Theorem 3.1.6

3.3.1 An inequality by Talagrand

Let S be a finite set and n a positive integer. Assign probabilities p_s, $s \in S$ to the elements of S. Let μ be the corresponding product measure on $\Omega := S^n$.

Let f be a function on Ω, and let $\|f\|_1$ and $\|f\|_2$ denote the L_1-norm and L_2-norm of f w.r.t. the measure μ:

$$\|f\|_1 := \sum_{x \in \Omega} \mu(x) |f(x)|;$$

$$\|f\|_2 := \sqrt{\sum_{x \in \Omega} \mu\,(x)\ |f\,(x)\,|^2}.$$

The notation $\bar{f}_i$ is used for the conditional expectation of f given all coordinates except the ith. More precisely, for $x = (x_1, \cdots, x_n) \in S^n$ we define

$$\bar{f}_i(x) := \sum_{s \in S} p_s\, f(x_1, \cdots, x_{i-1}, s, x_{i+1}, \cdots, x_n).$$

Further, we define the function $\Delta_i f$ on Ω by

$$(\Delta_i f)(x) = f(x) - \bar{f}_i(x),\ x \in \Omega. \tag{3.3.1}$$

*N*otational Remark: Often we work with the alternative, equivalent, description that we have n independent random variables, say $Z_1, \cdots, Z_n$, with $P(Z_i = s) = p_s,\ s \in S, 1 \leq i \leq n$. To emphasize the identity of the random variables involved, we then often use the notation Δ_{Z_i} instead of Δ_i.

A key ingredient in [13] and in our paper is the following inequality for the case $|S| = 2$ by Talagrand, a far-reaching extension of an inequality by Kahn, Kalai and Linial [62].

Theorem 3.3.1. [Talagrand [90], Theorem 1.5)]
There is a constant $K > 0$ such that for each n and each function f on $\{0,1\}^n$,

$$\mathrm{Var}(f) \leq K \log\left(\frac{2}{p(1-p)}\right) \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(e\, \|\Delta_i f\|_2 \,/\, \|\Delta_i f\|_1\right)}, \tag{3.3.2}$$

where (in the notation in the beginning of this section) $p = p_1 = 1 - p_0$, and where Var(f) denotes the variance of f w.r.t. the measure μ.

In the literature, (partial) extensions of this inequality and inequalities of related flavour, to the case $|S| > 2$ have been given; see e.g. [82] and [12]. The following Theorem (see [69]) states the most 'literal' extension of Theorem 3.3.1 to the case $|S| > 2$. (In [69], an extended version of Beckner's inequality, a key ingredient in the proof of Theorem 3.3.1, is used, and the proof of Talagrand is followed, with appropriate adaptations, to obtain the extension of Theorem 3.3.1). To make comparison of our line of arguments with that in [13] as clear as possible, it is this extension we will use. (Moreover, if instead of Theorem 3.3.2 we would use the modified Poincaré inequalities in [12], this would not simplify our proof of Theorem 3.1.6).

Theorem 3.3.2. [[69], Theorem 1.3] *There is a constant $K > 0$ such that for each finite set S, each $n \in \mathbb{N}$ and each function f on S^n the following holds:*

$$\mathrm{Var}(f) \leq K \log\left(\frac{1}{\min_{s\in S} p_s}\right) \sum_{i=1}^{n} \frac{\|\Delta_i f\|_2^2}{\log\left(e\, \|\Delta_i f\|_2 \,/\, \|\Delta_i f\|_1\right)}. \tag{3.3.3}$$

3.3.2 Greedy lattice animals

The subject of this subsection played no role in the treatment of the first-passage percolation model with independent t-variables in [13], but turns out to be important in our treatment of dependent t-variables.

Consider, for $d \geq 2$, the d-dimensional cubic lattice. A lattice animal (abbreviated as l.a.) is a finite connected subset of $\mathbb{Z}^d$ containing the origin. Let $X_v, v \in \mathbb{Z}^d$, be i.i.d. non-negative random variables with common distribution F. Define

$$N(n) := \max_{\zeta\,:\,\zeta \text{ l.a. with } |\zeta|=n} \sum_{v\in\zeta} X_v,$$

where the maximum is over all lattice animals of size n.

The subject was introduced by Cox, Gandolfi, Griffin and Kesten (1993) [33]. The asymptotic behaviour, as $n \to \infty$ of $N(n)$ has been studied in that and several other papers (see e.g. [43] and [58]). For our purpose the following result by Martin [75] is very suitable:

Theorem 3.3.3. [Martin ([75], Theorem 2.3)]
There is a constant C such that for all n and for all F that satisfy

$$\int_0^\infty (1 - F(x))^{1/d}\, dx < \infty,$$

$$E\left(\frac{N(n)}{n}\right) \leq C \int_0^{\infty} (1 - F(x))^{1/d}\, dx. \tag{3.3.4}$$

The paper [75] says considerably more than this, but the above is sufficient for our purpose.

3.3.3 A randomization tool

As in [13] we need, for technical reasons, a certain 'averaging' argument: extra randomness is added to the system to make it more tractable. To handle this extra randomness appropriately, the following Lemma from [13] is used:

Lemma 3.3.4. [Benjamini, Kalai, Schramm ([13], Lemma 3)] *There is a constant $c > 0$ such that for every $m \in \mathbb{N}$ there is a function*

$$g = g_m : \{0,1\}^{m^2} \to \{0, 1, \ldots, m\}$$

which satisfies properties (i) and (ii) below:
(i) For all $i = 1, \cdots, m^2$ and all $x \in \{0,1\}^{m^2}$,

$$|g_m(x^{(i)}) - g_m(x)| \leq 1, \tag{3.3.5}$$

where $x^{(i)}$ denotes the element of $\{0,1\}^{m^2}$ that differs from x only in the ith coordinate.
(ii)

$$\max_k \mathbb{P}(g(y) = k) \leq c/m, \tag{3.3.6}$$

where y is a random variable uniformly distributed on $\{0,1\}^{m^2}$.

3.4 Proof of Theorem 3.1.6

To keep our formulas compact, we will use constants C_1, C_2, $\cdots$. The precise values of these constants do not matter for our purposes. Some of them depend on a, b, the dimension d, the distribution of the X-variables (in terms of which the t-variables are represented), or the constants in the Conditions (i), (i'), (ii), (iii) and (iv) in Section 3.1.4. However, they do not (and obviously should not) depend on the choice of v in the statement of the theorem.

3.4.1 Informal sketch

The detailed proof is given in the next Subsection. Now we first give a very brief and rough summary of the proof of the main result in [13] (listed as Theorem 3.1.1 in our paper), and then informally (and again briefly) indicate the extra problems that arise in our situation where the t-variables are dependent.

Let γ be the path from 0 to v for which the sum of the t-variables is minimal. (If more than one such path exists, choose one of these by a deterministic procedure). Since the value of each t-variable is at least $a > 0$ and at most b, it is clear that the number of edges of γ is at most a constant c times $|v|$.

In [13] the t-variables are independent, and Talagrand's inequality (Theorem 3.3.1) is applied with $f = T(0, v)$ and with each i denoting an edge e. From the definitions it is clear that $\Delta_i f$ is roughly the change of $T(0, v)$ caused by changing $t(e)$. Moreover, a change of $t(e)$ can only cause a change of $T(0, v)$ if, before or after the change, e is on the above mentioned path γ. So, ignoring the denominator in Talagrand's inequality, one gets the (linear) bound

$$\mathrm{Var}(T(0, v)) \leq C_1\, \mathbb{E}\left[\sum_{e\in\gamma}(b-a)^2\right] \leq c(b-a)^2|v|. \tag{3.4.1}$$

It turns out that, by introducing additional randomness in an appropriate way, without changing the variance (see Lemma 3.3.4), the $\|\Delta_i f\|_2 \,/\, \|\Delta_i f\|_1$ in the denominator in the r.h.s. of Talgrand's inequality becomes (uniformly in i) larger than $|v|^\beta$ for some $\beta > 0$, thus giving the $\log|v|$ (and hence the sublinearity) in Theorem 3.1.1.

In our situation, the underlying independent random variables are the $X_i, i \in I$ (by which the *dependent* t-variables are represented). Application of Talagrand-type inequalities to these variables has the complication that changing one X-variable changes a (random) set of possibly many t-variables. Taking the square of the effect complicates this further. Nevertheless, it turns out that by suitable decompositions of the summations, and by block arguments (rescaling), one finally gets, instead of (3.4.1) a bound in terms of ('rescaled') greedy lattice animals which, by the result of Martin in Section 3.3, is still linear in $|v|$.

To handle the denominator in the Talagrand-type inequality, we use additional randomness, as in [13]. Again, the fact that changing an X variable can have effect on many t-variables complicates the analysis, but this complication is easier to handle than that for the numerator mentioned above.

3.4.2 Detailed proof

We give the proof for the site version of Theorem 3.1.6. The proof for the bond version is obtained from it by a straight-forward, step-by-step translation.

Notational remark: the cardinality of a set V will be indicated by $|V|$.

We start by stating a simple but important observation (a version of which was also used in [13]). A finite path π is called an optimal path, or a geodesic, if there is no path $\pi' \neq \pi$ with the same starting and endpoint as π, for which $T(\pi') < T(\pi)$.

Observation 3.4.1. *Since the t-variables are bounded away from 0 and ∞, there is a constant $C_2 > 0$ such that for every positive integer n and every $w \in \mathbb{Z}^d$ the following holds:*
(a) Each geodesic has at most $C_2 n$ vertices in the box $w + [-n, n]^d$.
(b) Each geodesic which starts at 0 and ends at w has at most $C_2|w|$ vertices.

Let $X_i, i \in I$, be the independent random variables in terms of which the variables $(t(v),\, v \in \mathbb{Z}^d)$ are represented. So $T(0,v)$ is a function of the X-variables. As we said in the informal sketch, we introduce extra randomness, in the same way as in [13]: Fix $m := \lfloor |v|^{1/4} \rfloor$. Let $(y_i^j,\, i = 1, \cdots, m^2,\, j = 1, \cdots, d)$ be a family of independent random variables, each taking value 0 or 1 with probability 1/2. The family of y_i^j's is also taken independently of the X variables. Define, for $j = 1, \cdots, d$,

$$y^j = (y_1^j, \cdots, y_{m^2}^j).$$

Each y^j is uniformly distributed on $\{0,1\}^{m^2}$, and will play the role of the y in Lemma 3.3.4. We simply write Y for the collection $(y_i^j,\, i = 1, \cdots, m^2,\, j = 1, \cdots, d)$ and X for the collection $(X_i, i \in I)$.
Let

$$z(Y) = \left(g(y^1), \cdots, g(y^d)\right), \tag{3.4.2}$$

with $g = g_m$ as in Lemma 3.3.4.

To shorten notation we will write f for $T(O,v)$ and $\tilde{f}$ for the passage time between the vertices that are obtained from 0 and v by a (random) shift over the vector $z(Y)$:

$$\tilde{f} = T(z(Y), v + z(Y)). \tag{3.4.3}$$

Note that f is completely determined by X, while $\tilde{f}$ depends on X as well as Y.

By translation invariance (see Condition (iii)), for every $w \in \mathbb{Z}^d$, $T(0,v)$ has the same distribution as $T(w, v+w)$. Hence, by conditioning on Y and using that Y is independent of the t variables, it follows that $\tilde{f}$ has the same distribution as f. In particular,

$$\mathrm{Var}(f) = \mathrm{Var}(\tilde{f}). \tag{3.4.4}$$

Theorem 3.3.2 gives (see the Remarks below):

$$\begin{aligned} \mathrm{Var}\left(\tilde{f}\right) \quad &\leq \quad C_3 \sum_{i=1,\ldots,m^2, j=1,\cdots,d} \frac{\left\|\Delta_{y_i^j} \tilde{f}\right\|_2^2}{1 + \log\left(\left\|\Delta_{y_i^j} \tilde{f}\right\|_2 \,/\, \left\|\Delta_{y_i^j} \tilde{f}\right\|_1\right)} \\ &\quad + C_3 \frac{\sum_{i \in I} \left\|\Delta_{X_i} \tilde{f}\right\|_2^2}{1 + \min_{i \in I} \log\left(\left\|\Delta_{X_i} \tilde{f}\right\|_2 \,/\, \left\|\Delta_{X_i} \tilde{f}\right\|_1\right)}. \end{aligned} \tag{3.4.5}$$

Remarks. (a) At first sight Theorem 3.3.2 is not applicable in the current situation where we have two types of random variables: X_i's and y_i^j's. However, by a straightforward argument, 'pairing' each variable y_i^j, $i = 1, \cdots, m^2$,

$j = 1, \cdots, d$, with an independent 'dummy' variable X_i^j (with the same distribution as the 'ordinary' X variables), and each variable X_i, $i \in I$, with an independent 'dummy' variable y_i (with the same distribution as the 'ordinary' y variables), it is easy to see that Theorem 3.3.2 is indeed applicable here.
(b) Note that the statement of Theorem 3.3.2 is formulated for finite n. Combined with a standard limit argument it gives (3.4.5).

We will handle, in separate subsections, the first term of (3.4.5), the numerator of the second term, and the denominator of the second term.

The first term in (3.4.5)

By (3.4.2), (3.4.3) and (3.3.5) it follows that $|\Delta_{y_i^j} \tilde{f}|$ is at most a constant C_4, so that we have the following lemma.

Lemma 3.4.2. *The first term in* (3.4.5) *is at most*

$$\leq dC_4 m^2 = dC_4 \left|v\right|^{1/2}. \tag{3.4.6}$$

The denominator of the second term in (3.4.5)

In this subsection we write, for notational convenience, $\Delta_i \tilde{f}$ for $\Delta_{X_i} \tilde{f}$, where $i \in I$.

If $w, w' \in \mathbb{Z}^d$ we write $\gamma_{w,w'}$ for the path π minimizing $\sum_{w \in \pi} t(w)$. If there is more than one such path, we use a deterministic, translation-invariant way to select one. If $w = 0$ and w' is our 'fixed' v, we write simply γ for $\gamma_{0,v}$.
Recall that $z = z(Y)$ is the random shift. We write $\gamma(z)$ for $\gamma_{z,v+z}$.

Also recall the definitions and notation in Section 3.1.4. If $w \in \mathbb{Z}^d$ and $j \in I$, we say that w needs j if $j = i_k(w)$ for some positive integer k, and $X_{i_1(w)}, \cdots, X_{i_{k-1}(w)}$ does not determine $t(w)$.

By a well-known second-moment argument we have, for each $j \in I$,

$$\frac{\left\|\Delta_j \tilde{f}\right\|_2}{\left\|\Delta_j \tilde{f}\right\|_1} \geq \frac{1}{\sqrt{\mathbb{P}\left(\Delta_j \tilde{f} \neq 0\right)}}. \tag{3.4.7}$$

Note that, given $z(Y)$ and all X_i, $i \in I \backslash \{j\}$, there is a, possibly non-unique, $s = s(j, X, Y) \in S$ such that $\tilde{f}$ (now considered as a function of X_j only) takes its smallest value at $X_j = s$. Further note that if $\Delta_j \tilde{f} \neq 0$ then, after replacing the value of X_j by s, we have $\Delta_j \tilde{f} < 0$. So we get

$$\mathbb{P}\left(\Delta_j \tilde{f} < 0\right) \geq \mathbb{P}\left(\Delta_j \tilde{f} \neq 0\right) \min_{r \in S} \mathbb{P}(X_j = r),$$

and hence

$$\mathbb{P}\left(\Delta_j \tilde{f} \neq 0\right) \leq \frac{\mathbb{P}\left(\Delta_j \tilde{f} < 0\right)}{\min_{r \in S} \mathbb{P}(X_j = r)}. \tag{3.4.8}$$

Moreover, it follows from the definitions that if $\Delta_j \tilde{f} < 0$, there is a w on $\gamma(z)$ such that a certain change of X_j causes a change of $t(w)$. By this and (3.4.8), we have

$$\begin{aligned} \mathbb{P}(\Delta_j \tilde{f} \neq 0) &\leq C_5 \sum_{w \in \mathbb{Z}^d} \mathbb{P}(w \in \gamma(z),\, w \text{ needs } j) \\ &\leq C_5 \sum_{w \in \mathbb{Z}^d} \min\left(\mathbb{P}\left(w \in \gamma\left(z\right)\right), \mathbb{P}\left(w \text{ needs } j\right)\right). \end{aligned} \tag{3.4.9}$$

Recall the definition of m in the paragraph following Observation 3.4.1. Let $w \in \mathbb{Z}^d$ and consider the box $B_m(w) := w + [-m, m]^d$. We have

$$\mathbb{P}(w \in \gamma(z)) = \mathbb{P}(w - z \in \gamma).$$

By the construction of z, and (3.3.5), $w - z$ takes values in the above mentioned box $B_m(w)$. Also by the construction of z, and (3.3.6), each vertex of the box has probability $\leq C_6/m^d$ to be equal to $w - z$. Moreover, by Observation 3.4.1 at most $C_7 m$ vertices in the box are on γ. Hence, since γ is independent of z, it follows (by conditioning on γ) that

$$\mathbb{P}(w \in \gamma(z)) \leq C_7 m \,\frac{C_6}{m^d} \leq C_8 |v|^{-(d-1)/4}. \tag{3.4.10}$$

Further, by Condition (i), we have

$$\mathbb{P}(w \text{ needs } j) \leq \frac{c_0}{r_w(j)^{3d+\varepsilon_0}}, \tag{3.4.11}$$

where $r_w(j)$ (which we call the rank of j) is the positive integer k for which $i_k(w) = j$.

By (3.4.9), (3.4.10) and (3.4.11), we have, for every K,

$$\mathbb{P}(\Delta_j \tilde{f} \neq 0) \leq C_9 \left(|v|^{-(d-1)/4} \left|\{w \,:\, r_w(j) < K\}\right| + \sum_{k=K}^{\infty} \frac{|\{w \,:\, r_w(j) = k\}|}{k^{3d+\varepsilon_0}} \right). \tag{3.4.12}$$

Now, Condition (ii) implies, for each $j \in I$ and each $k > 0$,

$$|\{w \,:\, r_w(j) < k\}| \leq C_{10} k^d. \tag{3.4.13}$$

Hence, the first term between the brackets in (3.4.12) is at most

$$C_{10} |v|^{-(d-1)/4} K^d. \tag{3.4.14}$$

Further, using again (3.4.13) (and summation by parts) the sum over k in (3.4.12) is at most

$$C_{11} \sum_{k=K}^{\infty} \frac{k^d}{k^{3d+\varepsilon_0+1}} \leq C_{12} K^{-2d-\varepsilon_0}. \tag{3.4.15}$$

Combining (3.4.12), (3.4.14) and (3.4.15) we get

$$\mathbb{P}(\Delta_j \tilde{f} \neq 0) \leq C_{13} \left(|v|^{-(d-1)/4} K^d + K^{-2d-\varepsilon_0} \right). \qquad (3.4.16)$$

Now take for K the smallest positive integer satisfying $K^d \geq |v|^{(d-1)/8}$ and insert this in (3.4.16). This gives

$$\mathbb{P}(\Delta_j \tilde{f} \neq 0) \leq C_{14} |v|^{-(d-1)/8}, \qquad (3.4.17)$$

which together with (3.4.7) yields the following Lemma.

Lemma 3.4.3. *There is a constant $C_{15} > 0$ such that for all $v \in \mathbb{Z}^d$ the denominator of the second term in* (3.4.5) *is larger than or equal to*

$$C_{15} \log |v|.$$

The numerator of the second term in (3.4.5), and completion of the proof of Theorem 3.1.6

As in the previous subsection we write Δ_j for Δ_{X_j}, where $j \in I$.

By the definition of $\tilde{f}$ (and of the norm $\| \cdot \|_2$), we rewrite

$$\sum_{j \in I} \|\Delta_j \tilde{f}\|_2^2 = \sum_{j \in I} \mathbb{E} \left[\left(\Delta_j T \left(z\left(Y\right), z\left(Y\right) + v \right) \right)^2 \right], \qquad (3.4.18)$$

By taking the expectation outside the summation, conditioning on Y (and using that Y is independent of the t-variables) and then taking the expectation back inside the summation, it is clear that the r.h.s. of (3.4.18) is smaller than or equal to

$$\max_{x \in \mathbb{Z}^d} \sum_{j \in I} \mathbb{E} \left((\Delta_j T(x, x+v))^2 \right). \qquad (3.4.19)$$

We will give an upper bound for the sum in (3.4.19) for the case $x = 0$. From the computations it will be clear that this upper bound does not use the specific choice of x, and hence holds for all x.

In the case $x = 0$, the sum in (3.4.19) is, by definition, of course

$$\sum_{j \in I} \|\Delta_j f\|_2^2. \qquad (3.4.20)$$

Let X'_j be an auxiliary random variable that is independent of the X variables and has the same distribution. Let X denote the collection of random variables $(X_i, i \in I)$, and X' the collection obtained from the collection X by replacing X_j by X'_j. By the definition of $\Delta_j f$ (and standard arguments) we have

$$\begin{aligned}
\mathbb{E}_j\left((\Delta_j f)^2\right) &= \frac{1}{2}\mathbb{E}_{j,j'}\left[\left(f(X)-f(X')\right)^2\right] \qquad (3.4.21)\\
&= \mathbb{E}_{j,j'}\left[\left(f(X)-f(X')\right)^2 I\left(f(X)<f(X')\right)\right],
\end{aligned}$$

where $\mathbb{E}_j$ denotes the expectation with respect to X_j, and $\mathbb{E}_{j,j'}$ denotes the expectation with respect to X_j and X'_j. (So, (3.4.21) is a function of the collection $(X_i,\ i \in I, i \neq j)$).

Let γ be the optimal path, as defined in the beginning of Subsection 3.4.2, w.r.t. the t-variables corresponding with the family X. Let w be a vertex. Observe that a change of $t(w)$ does not increase f if w is not on γ, and increases f by at most $b-a$ if w is on γ. By this observation, and a similar argument as used for (3.4.9), we have

$$\left(f(X')-f(X)\right) I\left(f(X)<f(X')\right) \leq (b-a)\sum_{w\in\gamma} I\left(w \text{ needs } j\right), \qquad (3.4.22)$$

and hence

$$(f(X)-f(X'))^2\, I(f(X)<f(X')) \leq (b-a)^2 \sum_{u,w\in\gamma} I(u \text{ needs } j, w \text{ needs } j). \qquad (3.4.23)$$

Since $\|\Delta_j f\|_2^2$ is the expectation w.r.t. the $X_i, i\neq j$, of $\mathbb{E}_j((\Delta_j f)^2)$, we have, by (3.4.21) and (3.4.23), that

$$\|\Delta_j f\|_2^2 \leq (b-a)^2\, \mathbb{E}\left[\sum_{u,w\in\gamma} I(u \text{ needs } j, w \text{ needs } j)\right]. \qquad (3.4.24)$$

To bound the r.h.s. of (3.4.24), recall the definition (below (3.4.11)) of $r_w(j)$ (with $j \in I$ and $w \in \mathbb{Z}^d$), and note that, by Condition (i) in Section 3.1.4, we have, on an event of probability 1,

$$\begin{aligned}
&\sum_{u,w\in\gamma} I(u \text{ and } w \text{ need } j) \qquad (3.4.25)\\
=\ &\sum_{k=1}^{\infty}\sum_{u,w\in\gamma} I\big(u \text{ and } w \text{ need } j,\, \max(r_u(j), r_w(j)) = k\big)\\
\leq\ & 2\sum_{k=1}^{\infty}\sum_{u\in\gamma}\sum_{w\in\gamma} I\big(u \text{ and } w \text{ need } j,\, r_u(j)=k,\, r_w(j)\leq k\big)\\
\leq\ & 2\sum_{k=1}^{\infty}\sum_{u\in\gamma} I(u \text{ needs } j,\, r_u(j)=k)\ |\{w\in\gamma : r_w(j)\leq k\}|.
\end{aligned}$$

By Condition (ii), each of the vertices w in the last line of (3.4.25) is located in a hypercube of length $C_{16}\,k$ centred at u. By this and Observation 3.4.1, it follows that the number of w's in the last line of (3.4.25) is at most $C_{17}k$. So we have, with $C_{18} = 2C_{17}$,

$$\sum_{u,w\in\gamma} I(u \text{ and } w \text{ need } j) \leq C_{18} \sum_{k=1}^{\infty} k \sum_{u\in\gamma} I(u \text{ needs } j,\, r_u(j) = k),$$

which, together with (3.4.24), (and using the definition of $i_k(u)$) gives, after summing over j,

$$\sum_{j\in I} \|\Delta_j f\|_2^2 \leq C_{19} \sum_{k=1}^{\infty} k\,\mathbb{E}\left[\sum_{u\in\gamma} I(u \text{ needs } i_k(u))\right] \tag{3.4.26}$$
$$= C_{19} \sum_{k=1}^{|v|} k\,\mathbb{E}\left[\sum_{u\in\gamma} I(u \text{ needs } i_k(u))\right] + C_{19} \sum_{k>|v|} k\,\mathbb{E}\left[\sum_{u\in\gamma} I(u \text{ needs } i_k(u))\right].$$

The sum over $k > |v|$ in the r.h.s. of (3.4.26) can be bounded very easily as follows: By Observation 3.4.1(b), all vertices of γ are inside the box $[-C_2|v|, C_2|v|]^d$. Hence the above mentioned sum over $k > |v|$ is at most

$$C_{19} \sum_{k>|v|} k \sum_{u\in[-C_2|v|,C_2|v|]^d} \mathbb{P}(u \text{ needs } i_k(u)).$$

By Condition (i), and since the number of vertices u in this last expression is, of course, of order $|v|^d$, this expression is smaller than or equal to a constant times

$$|v|^d \sum_{k>|v|} k^{1-3d-\varepsilon_0},$$

which is smaller than a constant C_{20}. (It is even $o(|v|)$, but that is not relevant for our purpose).

To bound the sum over $k \leq |v|$ in the r.h.s. of (3.4.26), observe that, by Condition (ii), if a set $V \subset \mathbb{Z}^d$ is such that $|u - u'| \geq C_{21}k$ for all $u, u' \in V$ with $u \neq u'$, then the collection of random variables

$$\big(I\,(u \text{ needs } i_k\,(u))\,,\ u \in V\big)$$

is independent. With this in mind, we partition, for each k, $\mathbb{Z}^d$ in boxes

$$B_k(w) := [-\lceil C_{21}k\rceil, \lceil C_{21}k\rceil)^d + 2\lceil C_{21}k\rceil w,\ \ w \in \mathbb{Z}^d.$$

We will say that two boxes $B_k(w)$ and $B_k(u)$ are neighbours (where $u = (u_1, \cdots, u_d)$ and $w = (w_1, \cdots, w_d)$) if $\max_{1\leq i\leq d} |w_i - u_i| = 1$.

By Observation 3.4.1(a), γ has at most $C_{22}k$ vertices in each of these boxes. Hence, the sum over $k \le |v|$ in the r.h.s. of (3.4.26) is at most

$$C_{23} \sum_{k=1}^{|v|} k^2 \, \mathbb{E}\left[\sum_{w\,:\,(*)} I\big(\exists u \in B_k(w) \,\text{s.t.}\, u \text{ needs } i_k(u)\big) \right], \tag{3.4.27}$$

where $(*)$ indicates that we sum over all $w \in \mathbb{Z}^d$ with the property that γ has a vertex in $B_k(w)$ or in a neighbour of $B_k(w)$.

Next, partition $\mathbb{Z}^d$ in 2^d classes, as follows:

$$\mathbb{Z}_z := z + 2\mathbb{Z}^d, \;\; z \in \{0,1\}^d.$$

So (3.4.27) can be written as

$$C_{23} \sum_{k=1}^{|v|} k^2 \sum_{z \in \{0,1\}^d} \mathbb{E}\left[\sum_{w\,:\,(**)} I\big(\exists u \in B_k(z+2w) \,\text{s.t.}\, u \text{ needs } i_k(u)\big) \right], \tag{3.4.28}$$

where (**) indicates that we sum over all $w \in \mathbb{Z}^d$ with the property that γ has a point in $B_k(z+2w)$ or in a neighbour of $B_k(z+2w)$.

Now, for each $z \in \{0,1\}^d$, the set

$$\{w \in \mathbb{Z}^d \,:\, \gamma \text{ has a point in } B_k(z+2w) \text{ or a neighbour of } B_k(z+2w)\}$$

is a lattice animal, and has, for $k \le |v|$, by Observation 3.4.1(b), at most $C_{24}|v|/k$ elements.

So, from (3.4.26)-(3.4.28) we get

$$\begin{aligned} &\sum_{j \in I} \|\Delta_j f\|_2^2 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.4.29) \\ &\le C_{23} \sum_{k=1}^{|v|} k^2 \sum_{z \in \{0,1\}^d} \mathbb{E}\left[\max_{\mathcal{L}\,:\,|\mathcal{L}| \le C_{24}|v|/k} \sum_{w \in \mathcal{L}} I\big(\exists u \in B_k(z+2w) \;\text{ s.t. }\; u \text{ needs } i_k(u)\big) \right] \\ &+ C_{20}, \end{aligned}$$

where the maximum is over all lattice animals $\mathcal{L}$ with size $\le C_{24}|v|/k$.

Now for each z we have, by the observation below (3.4.26), that

$$\Big(I\left(\exists u \in B_k(z+2w) \text{ s.t. } u \text{ needs } i_k(u)\right), \; w \in \mathbb{Z}^d \Big)$$

is a collection of independent $0-1$ valued random variables. For each w, this random variable is 1 with probability less than or equal to

$$|B_k(z+2w)| \max_{u \in \mathbb{Z}^d} \mathbb{P}(u \text{ needs } i_k(u)) \le \frac{C_{25}k^d}{k^{3d+\varepsilon_0}}, \tag{3.4.30}$$

where we used Condition (i).

By (3.4.29), (3.4.30) and Theorem 3.3.3, we get

$$\begin{aligned}\sum_{j\in I}\|\Delta_j f\|_2^2 &\leq C_{20}+C_{26}\sum_{k=1}^{|v|}k^2\frac{|v|}{k}\left(\frac{k^d}{k^{3d+\varepsilon_0}}\right)^{\frac{1}{d}} \\ &\leq C_{20}+C_{26}|v|\sum_{k=1}^{\infty}k^2k^{-(3d+\varepsilon_0)/d} \\ &\leq C_{27}|v|.\end{aligned} \tag{3.4.31}$$

Together with (3.4.18)-(3.4.20), this gives the following Lemma.

Lemma 3.4.4. *The numerator of the second term in* (3.4.5) *is at most* $C_{27}|v|$.

Lemma 3.4.4, together with (3.4.4), (3.4.5), Lemma 3.4.2 and Lemma 3.4.3, completes the proof of Theorem 3.1.6.

3.5 Proof of Theorem 3.1.7

The proof is very similar to that of Theorem 3.1.6 and we only discuss those steps that need adaptation.

First, we define, for $u, w \in \mathbb{Z}^d$, the following modification of $T(u, w)$:

$$\hat{T}(u,w) := \min_{\pi:u\to w,\,|\pi|\leq c_1|u-w|} T(\pi), \tag{3.5.1}$$

where $|\pi|$ is the number of vertices of π, and with c_1 as in Condition (iv).

From this definition it is obvious that

$$|\hat{T}(0,v)-T(0,v)| \leq b\,(|v|+1)\,I(\nexists \text{ an optimal path } \pi \text{ from } 0 \text{ to } v \text{ with } |\pi| < c_1|v|).$$

By this inequality and Condition (iv), we get immediately

$$\mathrm{Var}(\hat{T}(v)) - \mathrm{Var}(T(v)) = \mathrm{o}(|v|/\log(|v|)),$$

so that it is sufficient to prove (3.1.12) for $\hat{T}(0,v)$.

Now, with $f = \hat{T}(0,v)$ and $\tilde{f} = \hat{T}(z, v+z)$ (with $z = z(Y)$ as in Section 3.4) the proof follows that of Theorem 3.1.6, with the following modifications:

A few lines above (3.4.10) we used that γ has at most C_7m vertices in the box $B_m(w)$. In the current situation we have to add, as a correction term, the probability that γ has more than C_7m vertices in that box. It follows easily from Condition (iv) that, with a proper choice of C_7, this probability goes to 0 faster than any power of m. Hence (recalling the definition of m) it is clear that (3.4.10) remains true. Therefore, the denominator of the second term in the proof of 3.1.6 is, in the current situation, again larger than a constant times $\log|v|$.

A few lines before (3.4.26) we applied Observation 3.4.1(a) (which used the fact that all t-values were larger than some positive a) to conclude that the number of vertices of γ in a certain box of length of order k is at most some constant times k. In the current situation we do not have this strong bound, but we can obviously conclude that this number is at most the total number of vertices in the box. Because of this, the k in (3.4.26) is, in our current situation, replaced by k^d.

A few lines above (3.4.27) we again used Observation 3.4.1(a). Again we have to replace a factor k by k^d. By this (and the previous remark) the k^2 in (3.4.27), and therefore also in (3.4.28) becomes k^{2d}.

By the definition of $\hat{T}$, the statement about the size of the lattice animal (a few lines above (3.4.29)) still holds (with appropriate constants). By this and the earlier remarks, we now get (3.4.29) with the factor k^2 replaced by k^{2d}. By Condition (i'), the denominator in the r.h.s. of (3.4.30) is now of order $\exp(\varepsilon_0 k^{\varepsilon_1})$, so that the sum over k in this modified form of (3.4.31) is still finite.

This completes the proof of Theorem 3.1.7

4 Frozen percolation on the binary tree

This chapter is based on the paper [96] with Jacob van den Berg and Pierre Nolin.

Abstract

We study a percolation process on the planted binary tree, where clusters freeze as soon as they become larger than some fixed parameter N. We show that as N goes to infinity, the process converges in some sense to the frozen percolation process introduced by Aldous in [4].

In particular, our results show that the asymptotic behaviour differs substantially from that on the square lattice, on which a similar process has been studied recently by van den Berg, de Lima and Nolin [94].

Key words and phrases. percolation, frozen cluster.
AMS 2000 subject classifications. Primary 60K35; Secondary 82B43.

4.1 Introduction and statement of results

Aldous [4] introduced a percolation process where clusters are frozen when they get infinite, which can be described as follows. Let $G = (V, E)$ be an arbitrary simple graph with vertex set V, and edge set E. On every edge $e \in E$, there is a clock which rings at a random time τ_e with uniform distribution on $[0, 1]$, these random times τ_e, $e \in E$, being independent of each other. At time 0, all the edges are closed, and then each edge $e = (u, v) \in E$ becomes open at time τ_e if the open clusters of u and v at that time are both finite – otherwise, e stays closed. In other words, an open cluster stops growing as soon as it becomes infinite: it freezes, hence the name *frozen percolation* for this process.

The above description is informal – it is not clear that such a process exists. In [4], Aldous studies the special cases where G is the infinite binary tree (where every vertex has degree three), or the planted binary tree (where one vertex, the root vertex, has degree one, and all other vertices have degree three). He showed that the frozen percolation process exists for these choices of G. See Section 1.4.1 for more details. Recall from Section 1.4.3 that for $G = \mathbb{Z}^2$ there is no process satisfying the aforementioned evolution. It seems that no simple condition on the graph G is known that guarantees the existence of the frozen percolation process.

To get more insight in the non-existence for $\mathbb{Z}^2$, a modification of the process was studied in [94]. In the modified process, an open cluster freezes as soon as it reaches size at least N, where N (a positive integer) is the parameter of the model. See Definition 4.1.3 below for the meaning of "size". Formally, the evolution of a frozen percolation process with parameter N is the following.

At time 0, every edge is closed. At time t, an edge $e = (u, v) \in E$ becomes open if $\tau_{(u,v)} = t$ *and* the open clusters of u and v at time t have size strictly smaller than N – otherwise, e stays closed. We call this modified process the N-parameter frozen percolation process. Note that replacing N by ∞ corresponds formally to Aldous' infinite frozen percolation process, therefore we sometimes refer to it as the ∞-parameter frozen percolation process.

The N-parameter frozen percolation process does exist on $\mathbb{Z}^2$ (and on many other graphs, including the binary tree), since it can be described as a finite-range interacting particle system. For general existence results of interacting particle systems, see for example Chapter 1 of [73]. Van den Berg, de Lima and Nolin [94] study the distribution of the final cluster size (i.e. the size of the cluster of a given vertex at time 1). They show that, for $\mathbb{Z}^2$, the final cluster size is smaller than N, but still of order of N, with probability bounded away from 0. In the light of the earlier mentioned fundamental difference (the existence versus the non-existence of the ∞-parameter frozen percolation process), it is natural to ask if the N-parameter process for the planted binary behaves, for large N, very differently from that on $\mathbb{Z}^2$. It turns out that this is indeed the case: We show that the N-parameter frozen percolation process for the planted binary tree converges (in some sense, see Theorem 4.1.1) to Aldous' process as the parameter goes to infinity. In particular, the probability that the final cluster has size less than N, but of order N, converges to 0 (see (4.1.1) below).

Before stating our main result, let us give some notation. We denote the planted binary tree by T, and by $\mathscr{C}$ the set of finite connected subgraphs of T. We denote the (open) cluster of the root vertex at time t (that is, the connected component of the root vertex formed by the open edges at time t) by $\mathcal{C}_t$. For $C \in \mathscr{C}$, we denote by $|C|$ the number of edges of C. We distinguish between different frozen percolation processes by using subscripts for the probability measures. We thus use $\mathbb{P}_N$ to denote the probability measure for the N-parameter frozen percolation process where the size of a cluster is measured by the number of its edges, while for the ∞-parameter frozen percolation process, we use the notation $\mathbb{P}_\infty$. Our main result is the following.

Theorem 4.1.1. *For the N-parameter frozen percolation process on the planted binary tree, where the size of a cluster is measured by its number of edges, we* *have, for all* $t \in [0,1]$,

$$\mathbb{P}_N(\mathcal{C}_t = C) \to \mathbb{P}_\infty(\mathcal{C}_t = C) \ \text{ as } N \to \infty$$

for all $C \in \mathscr{C}$. *Moreover, for all* $t \in [0,1]$,

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mathbb{P}_N(k \le |\mathcal{C}_t| < N) = 0, \tag{4.1.1}$$

and hence the probability that the open cluster of the root vertex is frozen also converges:

$$\mathbb{P}_N(N \le |\mathcal{C}_t|) \to \mathbb{P}_\infty(|\mathcal{C}_t| = \infty) \ \text{ as } N \to \infty.$$

The theorem above considers the case where size of a cluster is measured by the number of its edges. It can be extended to other notions of size. To state our more general result, we need to introduce some additional definitions.

Definition 4.1.2. *We say that a function* h *on the set of vertices of* T *into itself is a* homomorphism *if it maps any edge* (s,t), *with* s *closer to the root than* t, *to an edge* $(h(s), h(t))$, *with* $h(s)$ *closer to the root than* $h(t)$.

Definition 4.1.3. *A* good *size function of finite connected subgraphs of* T *is a function* $s : \mathscr{C} \to \mathbb{N}$, *which satisfies the following conditions:*

1. *Compatibility with homomorphisms. For all* $C \in \mathscr{C}$ *and injective homomorphisms* h *we have* $s(h(C)) = s(C)$.
2. *Finiteness. For all* $N \in \mathbb{N}$ *and for any vertex* v, *the set*
$$\{C \in \mathscr{C} \mid v \in C,\, s(C) \le N\}$$
is finite.
3. *Monotonicity. If* $C, C' \in \mathscr{C}$ *with* $C \subseteq C'$, *then* $s(C) \le s(C')$.
4. *Boundedness above by the volume. For all* $C \in \mathscr{C}$, *we have* $s(C) \le |C|$.

The conditions of Definition 4.1.3 are satisfied for most of the usual size functions such as the diameter (the length of the longest self-avoiding path in the connected subgraph) or the depth (the length of the longest self-avoiding path starting from the root of the connected subgraph).

We indicate the dependence on the size function with an additional superscript: $\mathbb{P}_N^{(s)}$ denotes the probability measure for the N-parameter frozen percolation process with size function s. With this notation, the following generalization of Theorem 4.1.1 holds.

Theorem 4.1.4. *Let* s *be a good size function for the planted binary tree. Then we have, for all* $t \in [0,1]$,

$$\mathbb{P}_N^{(s)}(\mathcal{C}_t = C) \to \mathbb{P}_\infty(\mathcal{C}_t = C) \ \text{ as } N \to \infty \tag{4.1.2}$$

for all $C \in \mathscr{C}$. Moreover, for all $t \in [0,1]$

$$\lim_{k\to\infty} \limsup_{N\to\infty} \mathbb{P}_N^{(s)} \left(k \leq s\left(\mathcal{C}_t\right) < N \right) = 0, \tag{4.1.3}$$

and hence the probability that the open cluster of the root vertex is frozen also converges:

$$\mathbb{P}_N^{(s)} \left(N \leq s\left(\mathcal{C}_t\right)\right) \to \mathbb{P}_\infty \left(\left|\mathcal{C}_t\right| = \infty \right).$$

Remark 4.1.5. Equation (4.1.2) is valid even without condition 4 of Definition 4.1.3.

Remark 4.1.6. The behaviour described in Theorem 4.1.4 is very different from that of the square lattice: In [94] it is showed that for $G = \mathbb{Z}^2$, where size of a connected subgraph is measured by the diameter (denoted by *diam*), for any fixed $a, b \in \mathbb{R}$ with $0 < a < b < 1$,

$$\liminf_{N\to\infty} \mathbb{P}_N^{(diam)} \left(aN < diam\left(\mathcal{C}_1\right) < bN \right) > 0, \tag{4.1.4}$$

while this probability tends to 0 when G is the planted binary tree, thanks to Eq.(4.1.3).

Let us finally mention that since Aldous' seminal paper [4], several related questions were studied. For example, Chapter 4 of [26] considers frozen percolation on $\mathbb{Z}$, and variants of that model are investigated in [80] and [16], respectively on the complete graph and on the uniform random tree. In [9] many other aspects, including the measurability (w.r.t the τ values) of the ∞-parameter frozen percolation process on the binary tree, were investigated.

The paper is organized as follows. In Section 4.2 we prove Theorem 4.1.1. The proof relies on a careful study of the probability that the root edge is closed at time t, which we denote by $\beta_N(t)$. In Sections 4.2.1 and 4.2.2 we show that β_N satisfies a first order differential equation which involves the generating function of the Catalan numbers. In Section 4.2.3, we give an implicit solution of the aforementioned differential equation, and we use this in Sections 4.2.4 and 4.2.5 to prove the convergence of β_N as $N \to \infty$. We finish the proof of Theorem 4.1.1 in Section 4.2.6. In Section 4.3 we point out the changes in the proof of Theorem 4.1.1 required to prove Theorem 4.1.4.

Acknowledgment

We thank the anonymous referees for very useful comments, in particular for a suggestion which led to a shorter proof of Proposition 4.2.6.

4.2 Proof of Theorem 4.1.1

4.2.1 Setting

In this section, we consider the N-parameter frozen percolation process on the planted binary tree (where one vertex, the root vertex, has degree one, and all the other vertices have degree three) where the size of a connected subgraph of T is measured by its number of edges. We recall the notation $\mathbb{P}_N$. We denote by $\mathcal{A}_t$ the set of open edges at time t.

Let $e_0 = (v_0, v_1)$ be the root edge, where v_0 is the root vertex. The central quantity of our analysis is the following probability:

$$\beta_N(t) := \mathbb{P}_N(e_0 \notin \mathcal{A}_t) = \mathbb{P}_N(e_0 \text{ is closed at time } t) \tag{4.2.1}$$

(note that $\beta_N(t) = \mathbb{P}_N(|\mathcal{C}_t| = 0)$). In particular, $\beta_N(0) = 1$.

Remark 4.2.1. From the definition, it is easy to see that $\beta_N(t)$ is decreasing in t. Moreover, from the equality

$$\beta_N(t) = 1 - t + \mathbb{P}_N(\tau_{e_0} < t \text{ but } e_0 \text{ is closed at time } t), \tag{4.2.2}$$

we can see that $(\beta_N(t) - 1 + t)$ is increasing in t.

For $e \in E$, $e \neq e_0$, $T \backslash \{e\}$ has two connected components, one which contains e_0, and one which does not. Let T_e denote the component which does not contain e_0, together with the edge e: T_e is a subtree of T, isomorphic to T.

For any edge e_1, we define the frozen percolation process on T_{e_1} in the following way. We consider the set of random variables τ_e, $e \in T_{e_1}$, and define the frozen percolation process on T_{e_1} in the same way as we did for T. We denote the set of open edges at time t by $\mathcal{A}_t(e_1)$. Note that the process $\mathcal{A}_t(e_1)$ has the same law as $\mathcal{A}_t$. Moreover, $\mathcal{A}_t(e_1)$ and $\mathcal{A}_t$ are coupled via the random variables τ_e, $e \in T_{e_1}$.

In the following, we think of clusters and connected subgraphs of T as sets of edges. The outer boundary of a $C \in \mathscr{C}$, denoted by ∂C, is the set of edges in $E \setminus C$ that have a common endpoint with one of the edges of C.

4.2.2 Differential equation for β_N

Let us denote the kth Catalan number by $c_k = \binom{2k}{k} / (k+1)$, and recall that the generating function of the Catalan numbers is (see for example Section 2.1 of [37])

$$C(x) = \sum_{k=0}^{\infty} c_k x^k = \frac{1 - \sqrt{1-4x}}{2x} = \frac{2}{1 + \sqrt{1-4x}},$$

which converges for $|x| \leq \frac{1}{4}$. We denote by C_N the Nth partial sum, that is

$$C_N(x) = \sum_{k=0}^{N} c_k x^k.$$

It turns out that the following partial sums are more convenient to use:

$$\tilde{C}_N(x) = \frac{C_N(x) - 1}{x} = \sum_{k=0}^{N-1} c_{k+1} x^k.$$

With this notation, we have:

Lemma 4.2.2. *β_N is differentiable, and its derivative satisfies*

$$\beta_N'(t) = -\beta_N(t)^2 \tilde{C}_N(t\beta_N(t)). \tag{4.2.3}$$

Remark 4.2.3. In the introduction we pointed out that the model exists, in particular the differential equation (4.2.3) with initial condition $\beta_N(0) = 1$ has a solution. On the other hand, the general theory of ordinary differential equations provides uniqueness.

Proof. Let us denote the open cluster of v_1 without the edge e_0 at time s by $\tilde{\mathcal{C}}_s$.

We use the defining evolution of the N-parameter frozen percolation process as follows: At time s, if $\tau_{e_0} = s$, then e_0 tries to become open, and it succeeds if and only if $\left|\tilde{\mathcal{C}}_s\right| \le N-1$. By conditioning on τ_{e_0}, we get that

$$\begin{aligned}\beta_N(t) &= 1 - \int_0^t \mathbb{P}_N\left(\left|\tilde{\mathcal{C}}_s\right| < N \mid \tau_{e_0} = s\right) ds \\ &= 1 - \int_0^t \sum_{k=0}^{N-1} \mathbb{P}_N\left(\left|\tilde{\mathcal{C}}_s\right| = k \mid \tau_{e_0} = s\right) ds.\end{aligned} \tag{4.2.4}$$

First we compute the probability $\mathbb{P}_N\left(\tilde{\mathcal{C}}_s = C \mid \tau_{e_0} = s\right)$ for $|C| \le N-1$. If $\tilde{\mathcal{C}}_s = C$, $|C| \le N-1$, then for all $e \in C$, e is open at time s. Moreover, for all $e' \in \partial C \setminus \{e_0\}$, e' is closed at time s. The latter event can happen in two ways: e' is closed at time s in its own frozen percolation process on $T_{e'}$, or there is a big cluster at time s in $T \setminus T_{e'}$ touching e'. Since $|C| < N$, on the event $\left\{\tilde{\mathcal{C}}_s = C, \tau_{e_0} = s\right\}$, the latter cannot happen. Hence

$$\left\{\tilde{\mathcal{C}}_s = C, \tau_{e_0} = s\right\} \subseteq \bigcap_{e' \in \partial C \setminus \{e_0\}} \{e' \notin \mathcal{A}_s(e')\} =: A.$$

Note that the event A and the random variables τ_e, $e \in C$ are independent. Moreover, conditionally on A, the events $e \in \mathcal{A}_s$, $e \in C$ are independent, and each of them has probability s, so that

$$\mathbb{P}_N\left(\tilde{\mathcal{C}}_s = C \,\middle|\, e' \notin \mathcal{A}_s(e') \text{ for } e' \in \partial C \setminus \{e_0\}, \tau_{e_0} = s\right) = s^{|C|}. \tag{4.2.5}$$

Recall that the processes $\mathcal{A}_s(e')$, $e' \in \partial C \setminus \{e_0\}$ are independent and have the same law as $\mathcal{A}_s$. Hence the events $e' \notin \mathcal{A}_s(e')$, $e' \in \partial C \setminus \{e_0\}$ are independent, and each of them has probability $\beta_N(s)$. This together with (4.2.5) gives that

$$\mathbb{P}_N\left(\tilde{\mathcal{C}}_s = C \mid \tau_{e_0} = s\right) = s^{|C|} \beta_N(s)^{|\partial C \setminus \{e_0\}|}.$$

Using that $\left|\partial \tilde{\mathcal{C}}_s \setminus \{e_0\}\right| = \left|\tilde{\mathcal{C}}_s\right| + 2$, we get

$$\mathbb{P}_N\left(\tilde{\mathcal{C}}_s = C \mid \tau_{e_0} = s\right) = \beta_N(s)^2 (s\beta_N(s))^{|C|}. \tag{4.2.6}$$

It is well known that the number of connected subgraphs $C \subseteq T$ having k edges which contain the vertex v_1 but not the edge e_0 is c_{k+1}, the $(k+1)$th Catalan number (see for example Theorem 2.1 of [37]). By this and (4.2.6) we can rewrite (4.2.4) as follows:

$$\begin{aligned}\beta_N(t) &= 1 - \int_0^t \beta_N(s)^2 \sum_{k=0}^{N-1} c_{k+1}\left(s\beta_N(s)\right)^k ds. \\ &= 1 - \int_0^t \beta_N(s)^2 \tilde{C}_N\left(s\beta_N(s)\right) ds. \end{aligned} \tag{4.2.7}$$

The integrand in (4.2.7) is bounded (since $s, \beta_N(s) \in [0,1]$ and $\tilde{C}_N$ are continuous). Thus we can differentiate Eq.(4.2.7), which completes the proof of Lemma 4.2.2. □

4.2.3 Implicit formula for β_N

Lemma 4.2.4 gives an implicit solution of (4.2.3) with initial condition $\beta_N(0) = 1$. Before stating and proving the proposition, let us give a heuristic computation to explain where that proposition comes from, without checking if the operations performed are legal or not.

Define the function $\gamma_N(t) = t\beta_N(t)$. It follows from Eq.(4.2.3) that γ_N satisfies

$$\frac{\gamma_N'(t)}{\gamma_N(t)\left(1-\gamma_N(t)\tilde{C}_N(\gamma_N(t))\right)} = \frac{1}{t},$$

so

$$\int_a^{\gamma_N(t)} \frac{dx}{x\left(1-x\tilde{C}_N(x)\right)} = \log t + b$$

for some constants a, b. Using $\int_a^{\gamma_N(t)} \frac{dx}{x} = \log t + \log\left(\beta_N(t)/a\right)$, we get

$$\int_a^{\gamma_N(t)} \frac{\tilde{C}_N(x)}{1-x\tilde{C}_N(x)} dx = -\log\beta_N(t) + b' \tag{4.2.8}$$

for another constant b'. Finally, by plugging in $\beta_N(0) = 1$ and $\gamma_N(0) = 0$, we can evaluate b', which gives

$$\int_0^{t\beta_N(t)} \frac{\tilde{C}_N(x)}{1-x\tilde{C}_N(x)} dx = -\log\beta_N(t).$$

This suggests the following lemma.

Lemma 4.2.4. *For $t \in [0,1]$, $\beta_N(t)$ is the unique positive solution of the equation in z*

$$\int_0^{tz} \frac{\tilde{C}_N(x)}{1-x\tilde{C}_N(x)} dx + \log z = 0, \tag{4.2.9}$$

with the constraint $tz < x_N$, where x_N is the unique positive solution of $x\tilde{C}_N(x) - 1 = 0$.

Proof. Let us fix N. First, the polynomial $x\tilde{C}_N(x) - 1 = C_N(x) - 2$ has a positive derivative for $x > 0$, it has thus exactly one non-negative root x_N, and this root has multiplicity one. Note that $x_N > 1/4$, since $C(x) > C_N(x)$ for $x \in (0, 1/4]$, and $C(1/4) = 2$. ($C_N(x)$ and $C(x)$ are close for large N, this also suggests that the root is close to $1/4$ for large N: we will indeed prove that in the following.)

Let us prove that for $t \in [0,1]$, there is exactly one non-negative solution of (4.2.9) with $tz < x_N$. As $x \nearrow x_N$, the integrand in (4.2.9) behaves like $\frac{\kappa}{x_N - x}$ for some positive constant κ (using that the positive root x_N of $x\tilde{C}_N(x) - 1$ has multiplicity one). Hence,

$$\int_0^{x_N} \frac{\tilde{C}_N(x)}{1 - x\tilde{C}_N(x)} dx = \infty. \tag{4.2.10}$$

On the other hand,

$$\int_0^{z} \frac{\tilde{C}_N(x)}{1 - x\tilde{C}_N(x)} dx < \infty$$

for $z \in [0, x_N)$. This shows that for every $t \in [0,1]$, there is exactly one positive real number $u_N(t)$ which satisfies the equation (4.2.9), and $tu_N(t) < x_N$.

To complete the proof of Lemma 4.2.4, it is enough to show that u_N is differentiable, that

$$u_N'(t) = -u_N(t)^2 \tilde{C}_N(tu_N(t)) \tag{4.2.11}$$

for $t \in [0,1]$, and that $u_N(0) = 1$. Indeed, as already noted in Remark 4.2.3, the differential equation (4.2.11) has a unique solution. A substitution into (4.2.9) shows that $u_N(0) = 1$. It is easy to check the conditions of the implicit function theorem, and get that $u_N(t)$ is a differentiable function with derivative satisfying

$$(tu_N'(t) + u_N(t)) \frac{\tilde{C}_N(tu_N(t))}{1 - tu_N(t)\tilde{C}_N(tu_N(t))} = -\frac{u_N'(t)}{u_N(t)},$$

from which simple computations give (4.2.11). This completes the proof of Lemma 4.2.4. □

4.2.4 Bounds on β_N

We now compare β_N with the corresponding function in Aldous' paper [4], where clusters are frozen as soon as they become infinite. In Aldous' model, one has

$$\beta_\infty(t) := \mathbb{P}_\infty(e_0 \text{ is closed at time } t) = \begin{cases} 1 - t & \text{if } t \in [0, 1/2], \\ \frac{1}{4t} & \text{if } t \in [1/2, 1]. \end{cases}$$

The following bounds hold true:

Lemma 4.2.5. *We have*

$$0 \le \beta_N(t) - \beta_\infty(t) \le 2(x_N - 1/4) \quad \text{for all } t \in [0,1],$$

where x_N $(> 1/4)$ is the unique positive root of the polynomial $x\tilde{C}_N(x) - 1$.

Proof. From Lemma 4.2.4, we know that $t\beta_N(t) < x_N$, which combined with the definition of β_∞ gives the desired upper bound for $t \in \left[\frac{1}{2}, 1\right]$. We also know (Remark 4.2.1) that $\beta_N(t) - 1 + t$ is non-negative and increasing. Hence,

$$0 \le \beta_N(t) - 1 + t \le \beta_N(1/2) - 1/2 \le 2(x_N - 1/4) \tag{4.2.12}$$

for $t \in \left[0, \frac{1}{2}\right]$, by using also the previously proven upper bound at $t = \frac{1}{2}$. We have thus established the desired lower and upper bounds for $t \in \left[0, \frac{1}{2}\right]$. In particular, for $t = \frac{1}{2}$, we obtain that $\beta_N(1/2) \ge 1/2$.

Now, let us note that $t\beta_N(t)$ is increasing: this is an easy consequence of two facts, that $\beta_N(t)$ is decreasing and that the integrand in the left hand-side of (4.2.9) is positive. Combined with the bound $\beta_N(1/2) \ge 1/2$, we get

$$\frac{1}{4} \le \frac{1}{2}\beta_N(1/2) \le t\beta_N(t),$$

from which the desired lower bound for $t \in \left[\frac{1}{2}, 1\right]$ follows readily. This completes the proof of Lemma 4.2.5. □

4.2.5 Convergence to β_∞

It follows from Lemma 4.2.5 that in order to prove uniform convergence of the functions β_N to β_∞, it is enough to prove that $x_N \to 1/4$ as $N \to \infty$. We prove a bit more, namely we give an upper bound on the rate of convergence.

Proposition 4.2.6. *There exists a constant K such that $x_N - \frac{1}{4} < \frac{K}{N}$ for all $N \ge 1$. In particular,*

$$0 \le \beta_N(t) - \beta_\infty(t) \le \frac{2K}{N} \quad \text{for all } t \in [0,1] \text{ and } N \ge 1,$$

so that $\beta_N \to \beta_\infty$ uniformly on $[0,1]$.

Proposition 4.2.6 follows from the following lemma.

Lemma 4.2.7. *There exist constants $a, b > 0$ such that*

$$\sqrt{N}\left(C_N\left(\frac{1}{4} + \frac{x}{4N}\right) - 2\right) \ge ax - b$$

for all integer $N \ge 1$ and $x \in [0, \infty)$.

Proof of Proposition 4.2.6. Let us take $K \in \mathbb{R}$, $K > 0$ such that $K > b/a$. Then by Lemma 4.2.7, we have that for $N \geq 1$,

$$\sqrt{N}\left(C_N\left(\frac{1}{4}+\frac{K}{4N}\right)-2\right) \geq aK - b > 0,$$

and so

$$\left(\frac{1}{4}+\frac{K}{4N}\right)\tilde{C}_N\left(\frac{1}{4}+\frac{K}{4N}\right)-1 = C_N\left(\frac{1}{4}+\frac{K}{4N}\right)-2 > 0.$$

For any fixed N, the function $x \mapsto \left(\frac{1}{4}+\frac{x}{4N}\right)\tilde{C}_N\left(\frac{1}{4}+\frac{x}{4N}\right)-1$ is increasing on $[0,\infty)$. Hence, $\frac{1}{4}+\frac{K}{4N} > x_N$, that is $x_N - \frac{1}{4} < \frac{K}{4N}$. ☐

Proof of Lemma 4.2.7. Using that

$$2 = C(1/4) = \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{k+1}4^{-k},$$

we get

$$\begin{aligned}
&\sqrt{N}\left(C_N\left(\frac{1}{4}+\frac{x}{4N}\right)-2\right)\\
&\qquad = \sqrt{N}\sum_{k=0}^{N}\frac{\binom{2k}{k}}{k+1}4^{-k}\left((1+x/N)^k-1\right) - \sqrt{N}\sum_{k=N+1}^{\infty}\frac{\binom{2k}{k}}{k+1}4^{-k}\\
&\qquad =: (A)-(B).
\end{aligned} \tag{4.2.13}$$

Stirling's formula gives that there are positive constants K_1, K_2 such that

$$K_1 k^{-3/2} \leq \frac{\binom{2k}{k}}{k+1}4^{-k} \leq K_2 k^{-3/2} \tag{4.2.14}$$

for all $k \geq 1$.

Using the lower bound in Eq.(4.2.14), we get for $x \in [0,\infty)$

$$\begin{aligned}
(A) = \sqrt{N}\sum_{k=0}^{N}\frac{\binom{2k}{k}}{k+1}4^{-k}\left((1+x/N)^k-1\right) &\geq \sqrt{N}\sum_{k=0}^{N}\frac{\binom{2k}{k}}{k+1}4^{-k}\frac{kx}{N}\\
&\geq \frac{K_1 x}{\sqrt{N}}\sum_{k=1}^{N}\frac{1}{\sqrt{k}} \geq ax
\end{aligned} \tag{4.2.15}$$

for some positive $a \in \mathbb{R}$. The upper bound in Eq.(4.2.14) provides an upper bound for (B):

$$(B) = \sqrt{N}\sum_{k=N+1}^{\infty}\frac{\binom{2k}{k}}{k+1}4^{-k} \leq K_2\sqrt{N}\sum_{k=N+1}^{\infty}k^{-3/2} \leq b \tag{4.2.16}$$

for some positive $b \in \mathbb{R}$.

Substituting (4.2.15) and (4.2.16) into (4.2.13) provides the desired lower bound, finishing the proof of the lemma. ☐

Remark 4.2.8. It is also possible to prove that the functions $\sqrt{N}\left(C_N\left(\frac{1}{4}+\frac{x}{4N}\right)-2\right)$ converge locally uniformly in $x \in \mathbb{R}$ as $N \to \infty$ to the function

$$F(x) = \frac{2}{\sqrt{\pi}}\left(\sqrt{x}\int_0^x \frac{e^y}{\sqrt{y}}dy - e^x\right).$$

4.2.6 Completion of the proof of Theorem 4.1.1

Recall the notation $\mathcal{C}_t$. Let $|C| < N$ be a fixed connected subgraph of T containing the root vertex. Note that for the ∞-parameter frozen percolation process,

$$\mathbb{P}_\infty(\mathcal{C}_t = C) = \beta_\infty(t)(t\beta_\infty(t))^{|C|}$$

for all $t \in [0,1]$. For $t \in [1/2, 1]$, this follows from Proposition 1 of [4]. For $t \in [0, 1/2)$, there are no frozen clusters in the ∞-parameter model at time t. Hence, the cluster of the root vertex is a percolation cluster with parameter t, which gives for $t \in [0, 1/2)$,

$$\mathbb{P}_\infty(\mathcal{C}_t = C) = t^{|C|}(1-t)^{|\partial C|} = \beta_\infty(t)(t\beta_\infty(t))^{|C|}$$

(since $|\partial C| = |C| + 1$).

By similar arguments as in the proof of Lemma 4.2.2, we have

$$\mathbb{P}_N(\mathcal{C}_t = C) = t^{|C|}\beta_N(t)^{|\partial C|} = \beta_N(t)(t\beta_N(t))^{|C|}. \tag{4.2.17}$$

Hence for any fixed finite connected subgraph $C \subseteq T$ containing the root vertex, we have, as $N \to \infty$,

$$\mathbb{P}_N(\mathcal{C}_t = C) = \beta_N(t)(t\beta_N(t))^{|C|} \to \beta_\infty(t)(t\beta_\infty(t))^{|C|} = \mathbb{P}_\infty(\mathcal{C}_t = C), \tag{4.2.18}$$

which gives the first part of Theorem 4.1.1.

An argument similar to the beginning of the proof of Lemma 4.2.2 gives that

$$\mathbb{P}_N(k \le |\mathcal{C}_t| < N) = \beta_N(t)\sum_{n=k}^{N-1}\frac{\binom{2n}{n}}{n+1}(t\beta_N(t))^n.$$

Lemma 4.2.4 and Proposition 4.2.6 then imply that $t\beta_N(t) < x_N \le \frac{1}{4} + \frac{K}{4N}$, hence (using again Eq.(4.2.14))

$$\begin{aligned}\mathbb{P}_N(k \le |\mathcal{C}_t| < N) &= \beta_N(t)\sum_{n=k}^{N-1}\frac{\binom{2n}{n}}{n+1}(t\beta_N(t))^n \\ &\le K_2\sum_{n=k}^{N-1} n^{-3/2}\left(1+\frac{K}{N}\right)^n \\ &\le K_2 e^K \sum_{n=k}^{\infty} n^{-3/2} \le \frac{K'}{\sqrt{k}}.\end{aligned}$$

It follows that

$$\lim_{k\to\infty}\limsup_{N\to\infty}\mathbb{P}_N\left(k\leq|\mathcal{C}_t|<N\right)=0, \tag{4.2.19}$$

which completes the second part of Theorem 4.1.1.

Now, using the trivial upper bound $\mathbb{P}_N\left(N\leq|\mathcal{C}_t|\right)\leq\mathbb{P}_N\left(k\leq|\mathcal{C}_t|\right)$ for $k\leq N$, we get

$$\begin{aligned}\limsup_{N\to\infty}\mathbb{P}_N\left(N\leq|\mathcal{C}_t|\right)&\leq\lim_{k\to\infty}\limsup_{N\to\infty}\mathbb{P}_N\left(k\leq|\mathcal{C}_t|\right)\\&=\lim_{k\to\infty}\mathbb{P}_\infty\left(k\leq|\mathcal{C}_t|\right)=\mathbb{P}_\infty\left(|\mathcal{C}_t|=\infty\right),\end{aligned} \tag{4.2.20}$$

where we used (4.2.18) for the first equality.

On the other hand, for all $k\in\mathbb{N}$, $k\leq N$, we have

$$\mathbb{P}_N\left(N\leq|\mathcal{C}_t|\right)=\mathbb{P}_N\left(k\leq|\mathcal{C}_t|\right)-\mathbb{P}_N\left(k\leq|\mathcal{C}_t|<N\right). \tag{4.2.21}$$

Hence, taking first the limit infimum as $N\to\infty$, and then the limit as $k\to\infty$, we get

$$\begin{aligned}&\liminf_{N\to\infty}\mathbb{P}_N\left(N\leq|\mathcal{C}_t|\right)\\&\qquad\geq\lim_{k\to\infty}\liminf_{N\to\infty}\mathbb{P}_N\left(k\leq|\mathcal{C}_t|\right)-\lim_{k\to\infty}\limsup_{N\to\infty}\mathbb{P}_N\left(k\leq|\mathcal{C}_t|<N\right)\\&\qquad=\mathbb{P}_\infty\left(|\mathcal{C}_t|=\infty\right)-0,\end{aligned} \tag{4.2.22}$$

where for the last equality we used, respectively, (4.2.18) – as in (4.2.20) – and (4.2.19).

Combining (4.2.20) and (4.2.22) provides the final part of Theorem 4.1.1.

□

4.3 Proof of Theorem 4.1.4

In this section we give a brief outline of the changes required to deduce Theorem 4.1.4 from the arguments in Section 4.2.

First, for any good size function s, the corresponding N-parameter frozen percolation process does exist. Indeed, conditions 1 and 2 of Definition 4.1.3 ensure that the process is still a finite-range interacting particle system, and the general theory of such systems [73] provides existence, as in the case where the size of connected subgraphs was measured by the number of edges.

In that previous case, the function $\tilde{C}_N\left(x\right)$ played an important role. It is the generating function of the number of connected subgraphs of T containing the vertex v_1 but not the edge e_0 and at most $N-1$ edges. For other good size functions s, the following generating function plays the role of $\tilde{C}_N\left(x\right)$:

$$G_N^{(s)}\left(x\right)=\sum_{k=0}^{\infty}a_{k,N-1}^{(s)}x^k,$$

where $a^{(s)}_{k,N-1}$ denotes the number of connected subgraphs $C \subseteq T$ containing v_1 for which $e_0 \notin C$, $|C| = k$ and $s(C) \leq N-1$.

Keeping this in mind, one can easily modify the proof of Theorem 4.1.1. We define the function $\beta^{(s)}_N : [0,1] \to \mathbb{R}$ as

$$\beta^{(s)}_N(t) := \mathbb{P}^{(s)}_N(e_0 \notin \mathcal{A}_t).$$

Using the conditions 1, 2 and 3 of Definition 4.1.3, by simple adjustments of the proof of Lemma 4.2.2 we deduce that $\beta^{(s)}_N$ is differentiable, and that its derivative satisfies

$$(\beta^{(s)}_N)'(t) = -\big(\beta^{(s)}_N(t)\big)^2 G^{(s)}_N\left(t\beta^{(s)}_N(t)\right).$$

Moreover, it follows from the definition of $\beta^{(s)}_N$ that $\beta^{(s)}_N(0) = 1$.

Recall that x_N, the unique positive root of $x\tilde{C}_N(x) = 1$, was another important quantity. Since in our present general setup $G^{(s)}_N(x)$ plays the role of $\tilde{C}_N(x)$, the analogue of x_N is the unique positive root of the equation $xG^{(s)}_N(x) = 1$, which we denote by $x^{(s)}_N$. Using the arguments of Section 4.2.3, we deduce that for each fixed t, $\beta^{(s)}_N(t)$ is equal to the unique positive root of the equation in z

$$\int_0^{tz} \frac{G^{(s)}_N(x)}{1 - xG^{(s)}_N(x)}dx + \log z = 0$$

with the constraint $tz < x^{(s)}_N$.

By simple modifications of Section 4.2.4, we get that $0 \leq \beta^{(s)}_N(t) - \beta_\infty(t) \leq 2\left(x^{(s)}_N - \frac{1}{4}\right)$ for all $t \in [0,1]$, which is the analogue of Lemma 4.2.5 in this general setting. By condition 3 of Definition 4.1.3, $a^{(s)}_{k,N-1}$ is an increasing function of N for each fixed k. Moreover, since $s(C)$ is finite for all finite connected subgraphs $C \subseteq T$, $a^{(s)}_{k,N-1} \uparrow c_{k+1}$ as $N \to \infty$. Hence $G^{(s)}_N(x) \uparrow \frac{C(x)-1}{x}$ for all $x \in \left[0, \frac{1}{4}\right]$, and $G^{(s)}_N(x) \uparrow \infty$ for $x > \frac{1}{4}$. Thus $x^{(s)}_N \to \frac{1}{4}$ as $N \to \infty$. By the aforementioned analogue of Lemma 4.2.5, we get that $\beta^{(s)}_N \to \beta_\infty$ point-wise. This concludes the proof of the first part (Eq.(4.1.2)) of Theorem 4.1.4.

Note that up to now we did not use that s satisfies Condition 4 of Definition 4.1.3. We use this condition to prove a rate of convergence for $x^{(s)}_N$, which was the key ingredient in the proof of (4.1.1). Condition 4 implies that $a^{(s)}_{k,N-1} \geq c_{k+1}$ for $k \leq N-1$, hence

$$G^{(s)}_N(x) \geq \tilde{C}_N(x) \text{ for } x \geq 0,$$

and thus $\frac{1}{4} \leq x^{(s)}_N \leq x_N = x^{(|\cdot|)}_N$. Proposition 4.2.6 then implies that $0 \leq x^{(s)}_N - \frac{1}{4} \leq \frac{K}{N}$, from which a computation similar to Section 4.2.6 completes the proof of Theorem 4.1.4.

5 Frozen percolation in two dimensions

This chapter is based on the paper [70].

Abstract

Aldous [4] introduced a modification of the bond percolation process on the binary tree where clusters stop growing (freeze) as soon as they become infinite. We investigate the site version of this process on the triangular lattice where clusters freeze as soon as they reach L^∞ diameter at least N for some parameter N. We show, informally speaking, that in the limit $N \to \infty$, the clusters only freeze in the critical window of site percolation on the triangular lattice. Hence the fraction of vertices that eventually (i. e. at time 1) are in a frozen cluster tends to 0 as N goes to infinity. We also show that the diameter of the open cluster at time 1 of a given vertex is, with high probability, smaller than N but of order N. This shows that the process on the triangular lattice has a behaviour quite different from Aldous' process. We also indicate which modifications have to be made to adapt the proofs to the case of the N-parameter frozen bond percolation process on the square lattice. This extends our results to the square lattice, and answers the questions posed by van den Berg, de Lima and Nolin in [77].

Keywords and phrases. frozen cluster, critical percolation, near critical percolation, correlation length.
AMS 2010 classifications Primary 60K35; Secondary 82B43.

5.1 Introduction

Stochastic processes where small fragments merge and form larger ones are quite useful tools to model physical phenomena at scales ranging from molecular [89] to astronomical ones [98]. The majority of the mathematical literature on such coagulation processes treats mean field models: The rate at which the fragments (clusters) merge is governed only by their sizes - neither the physical location nor their shape affect this rate. See [15] for a review. Stockmayer [89], introduced a mean field model for polymerization where small clusters (sol) merge, however, as soon as a large cluster (gel) forms, it stops growing. In contrast to the mean field models, we consider a model which takes the geometry of the space and the shape of the clusters into account. Following van den Berg, de Lima and Nolin [94], and Aldous [4], we introduce the following adaptation of Stockmayer's model. Let $G = (V, E)$ be a graph which represents the underlying geometry and $N \in \mathbb{N}$. For every vertex $v \in V$, independently from each other, we assign a random time τ_v which is uniformly distributed on $[0, 1]$. At time $t = 0$, all of the vertices of G are closed. As time increases, a vertex v tries to become open at time $t = \tau_v$. It succeeds if and only if all of its neighbours' open clusters (open connected components) at time t have size less than N. Note that as soon as the diameter of a cluster reaches N, it stops growing, i.e freezes. Hence the name N-parameter frozen percolation. Note that we can also consider an edge (bond) version of the model above where edges turn open from closed. This edge version of the process was introduced by van den Berg, de Lima and Nolin [94].

We are particularly interested in the N-parameter frozen percolation models for large N on graphs such as d dimensional lattices, since they are discrete approximations of the space $\mathbb{R}^d$. Herein we restrict to the case where $d = 2$. We mainly work on the triangular lattice. We will see that the behaviour of this model is rich and interesting too, but in a very different way from the model studied by Aldous [4].

Let us turn to the model introduced and constructed by Aldous [4]. It is the edge version of the model on the binary tree where we replace the parameter N by ∞ in the description above. An edge e of the binary tree opens at time τ_e as long as the open clusters of the endpoints of e are finite. In view of this model, one could also try to construct a similar, so called ∞-parameter, model on the triangular lattice. However Benjamini and Schramm [14] showed that it is impossible. See Section 1.4.3 for more details. Exactly this non-existence result motivated van den Berg, de Lima and Nolin [94] to extend the model of Aldous for finite parameter N: in this case, the N-parameter frozen percolation process (both the vertex and the edge version) is a finite range interacting particle system, hence the general theory [73] gives existence. One could ask if the N-parameter processes for large but finite N provide a reason for the existence of the ∞-parameter frozen bond percolation on the binary tree and the non-existence of the ∞-parameter frozen site percolation on the triangular lattice. Before we answer this question, let us specify the two dimensional model which plays a central role in this paper.

We work on the triangular lattice $\mathbb{T} = (V, E)$ with its usual embedding in the plane $\mathbb{R}^2$. That is, the vertex set V is the lattice generated by the vectors $\underline{e}_1 = (1, 0)$ and $\underline{e}_2 = (\cos(\pi/3), \sin(\pi/3))$:

$$V := \{a\underline{e}_1 + b\underline{e}_2 \,|a, b \in \mathbb{Z}\}. \tag{5.1.1}$$

The vertices u and v are neighbours, i.e $(u, v) \in E$ or $u \sim v$ if their L^2 distance is 1. We consider the model where we freeze clusters as soon as they reach L^∞

diameter (inherited from $\mathbb{R}^2$) at least N. For the case where the underlying lattice is $\mathbb{Z}^2$ and for different choices for diameters of clusters see the discussion below Conjecture 5.1.8.

In Chapter 4 we investigated the edge version of the N-parameter process on the binary tree. We found that as $N \to \infty$, the N-parameter process on the binary tree converges to the ∞-parameter process in some weak sense. This result raises the question if there is a limit of the N-parameter frozen percolation processes on the triangular lattice as N goes to infinity. The non-existence of the ∞-parameter process suggests that the N-parameter model may have a remarkable (anomalous) behaviour in the limit $N \to \infty$. It turns out that there is a limiting process, but this process is, in some sense, trivial:

Theorem 5.1.1. *As $N \to \infty$ the probability that in the N-parameter frozen percolation process the open cluster of the origin freezes goes to* 0.

To get some intuition for the behaviour of the process, let us for the moment forget about freezing, and call the resulting process the percolation process. That is, at time τ_v the vertex v becomes open no matter how big are the open clusters of its neighbours. Thus at time t, a vertex v is open with probability t independently from the other vertices. Hence at time t we see ordinary site percolation with parameter t. Recall from [84] that the critical parameter for site percolation on the triangular lattice is $p_c = 1/2$. So at each time $t \leq 1/2$ there is no open infinite cluster, and there is a unique infinite open cluster when $t > 1/2$. Moreover, by [3] at time $t < 1/2$, the distribution of the size of the open clusters has an exponential decay. Note that if a site is open in the N-parameter frozen percolation process at time t, then it is also open in the percolation process at time t. Hence at time $t < 1/2$ the N-parameter frozen percolation process and the percolation process does not differ too much when N is large: even without freezing, for all $K > 0$ the probability that there is an open cluster with diameter at least N in a box with side length KN goes to 0 as $N \to \infty$. To our knowledge, there is no simple argument showing that, roughly speaking, freezing does not take place at times that are essentially bigger than $1/2$, which is one of our main results:

Theorem 5.1.2. *For all $K > 0$ and $t > 1/2$, the probability that after time t a frozen cluster forms which intersects a given box with side length KN goes to* 0 *as $N \to \infty$.*

Compare Theorem 5.1.2 with [4] and Chapter 4 where it was shown that clusters freeze throughout the time horizon $[1/2, 1]$ for $N \in \mathbb{N} \cup \{\infty\}$ in the edge version of the N-parameter frozen percolation process on the binary tree. (Note that the critical parameter is $1/2$ for site percolation on the binary tree.) As it turns out, our method provides a much stronger result than Theorem 5.1.2. To state it we need some more notation.

Let $\mathbb{P}$ denote the probability measure corresponding to the percolation process. For a fixed $p \in [0, 1]$, we call a vertex $v \in V$ p-open (p-closed), if its τ value is less (greater) than p. We denote by $\mathbb{P}_p$ the distribution of p-open vertices.

We borrow some of the notation from [77]. Recall the definition of V from (5.1.1). The L^∞ distance of vertices in $\mathbb{T}$ is the L^∞ distance inherited from $\mathbb{R}^2$. That is, for $v, w \in V$ the distance $d(v,w)$ between $v = (v_1, v_2)$ and $w = (w_1, w_2)$ is

$$\begin{aligned} d(v,w) &= \|v-w\|_\infty \\ &= \max\{|v_1 - w_1|, |v_2 - w_2|\}. \end{aligned}$$

For $a, b, c, d \in \mathbb{R}$, with $a < b, c < d$ we define the parallelogram

$$[a,b] \boxtimes [c,d] := \{k\underline{e}_1 + l\underline{e}_2 \,|\, k \in [a,b] \cap \mathbb{Z},\ l \in [c,d] \cap \mathbb{Z}\}.$$

We denote the outer boundary of a set of vertices $S \subseteq V$ by

$$\partial S := \{v \in V \setminus S \,|\, \exists u \in S :\ u \sim v\}. \tag{5.1.2}$$

Let $cl(S) = S \cup \partial S$ denote the closure of S. For the parallelogram centred around the vertex v with radius $a > 0$ we write

$$B(v;a) := [-a,a] \boxtimes [-a,a] + v.$$

We denote the annulus centred around $v \in V$ with inner radius $a > 0$ and outer radius $b > a$ by

$$A(v;a,b) := B(v;b) \setminus B(v;a).$$

We call $B(v;a)$ the inner, $B(v;b)$ the outer parallelogram of $A(v;a,b)$.

We say that there is an open (closed) arm in an annulus $A(v;a,b)$ if there is an open (closed) path from $\partial B(v;a)$ to $\partial B(v;b)$ in $A(v;a,b)$. We write o for open and c for closed. A colour sequence of length k is an element of $\{o,c\}^k$. For $\sigma \in \{o,c\}^k$, we denote by $\mathcal{A}_{k,\sigma}(v;a,b)$ the event that there are k disjoint arms in $A(v;a,b)$ such that the vertices of each of the arms are either all open or all closed, moreover, if we take a counter-clockwise ordering of these arms, then their colours follow a cyclic permutation of σ.

In the case where $v = \underline{0} = (0,0)$ we omit the first argument in our notation, that is $B(a) = B(\underline{0};a)$ etc. For the critical arm probabilities we use the notation

$$\pi_{k,\sigma}(a,b) := \mathbb{P}_{1/2}(\mathcal{A}_{k,\sigma}(a,b)). \tag{5.1.3}$$

In the following we use the near critical parameter scale (1.1.7) of Section 1.1.3. It was introduced in [46]. For a positive parameter N and $\lambda \in \mathbb{R}$ it is defined as

$$p_\lambda(N) := \frac{1}{2} + \lambda \frac{N^{-2}}{\pi_{4,alt}(1,N)} \tag{5.1.4}$$

where alt denotes the colour sequence (o,c,o,c).

Before we proceed, let us stop here and let us briefly explain the formula (5.1.4). Suppose that a vertex v is a closed pivotal vertex, i.e. it is on the boundary of two different open cluster with diameter at least N. The two open clusters provide two disjoint open arms starting from neighbouring vertices of v. Since the open clusters are different, they have to be separated by closed paths, which provide two disjoint closed arms starting from v. Hence the event $\mathcal{A}_{4,alt}(v;1,N)$ occurs. By (5.1.3), we get that the expected number of pivotal vertices in $B(N)$ is $O\left(N^2\pi_{4,alt}(1,N)\right)$. Let $\lambda > 0$. Let us look at the percolation process in the parallelogram $B(N)$ in the time interval $[1/2, p_\lambda(N)]$. The probability that a vertex opens in this time interval is $p_\lambda(N) - 1/2$. By a combination of (5.1.3) and (5.1.4) we see that the expected number of pivotal vertices which open in this interval is $O(1)$. Hence the parameter scale in (5.1.4) corresponds to the time scale where open clusters of diameter $O(N)$ merge. See [45, 46] for more details.

The considerations above suggest that the parameter scale (5.1.4) is indeed useful for investigating the N-parameter frozen percolation process. We write $\mathbb{P}_N$ for the probability measure corresponding to the N-parameter frozen percolation process. The following stronger version of Theorem 5.1.2 is our main result.

Theorem 5.1.3. *For any $\varepsilon, K > 0$ there exists $\lambda = \lambda(\varepsilon, K)$ and $N_0 = N_0(\varepsilon, K)$ such that*

$$\mathbb{P}_N\left(\textit{a cluster intersecting } B(KN) \textit{ freezes after time } p_\lambda(N)\right) < \varepsilon$$

for all $N \geq N_0$.

In [94] the authors investigated the diameter of the open cluster of the origin at time 1. Their main result is the following.

Definition 5.1.4. *For $t \in [0,1]$ let $C(v;t)$ denote the open cluster of $v \in V$ at time $t \in [0,1]$. We set $C(t) := C(\underline{0};t)$.*

Definition 5.1.5. *For $C \subset V$, let $\operatorname{diam}(C)$ denote the L^∞-diameter of C.*

Theorem 5.1.6 (Theorem 1.1 of [94])**.** *For the bond version of the N-parameter frozen percolation on the square lattice we have*

$$\liminf_{N\to\infty} \mathbb{P}_N\left(\operatorname{diam}(C(1)) \in (aN, bN)\right) > 0$$

for $a, b \in (0,1)$ with $a < b$.

Analogous result holds for the (site version of) N-parameter process on the triangular lattice. In the following corollary we supplement this result. It is an extension of Theorem 5.1.1.

Corollary 5.1.7. *For any $\varepsilon > 0$ there exists $a = a(\varepsilon), b = b(\varepsilon) \in (0,1)$ with $a < b$ and $N_0 = N_0(\varepsilon)$ such that*

$$\mathbb{P}_N\left(\operatorname{diam}(C(1)) \in (aN, bN)\right) > 1 - \varepsilon$$

for all $N \geq N_0$.

The results above suggest the following intuitive and informal description of the behaviour of N-parameter frozen percolation processes on the triangular lattice for large N: At time 0 all the vertices are closed. Then they open independently from each other as in the percolation process till time close to $1/2$. Then in the scaling window (5.1.4), frozen clusters form, and by the end of the window, they give a tiling of $\mathbb{T}$ such that all the holes (non-frozen connected components) have diameter less than N but, typically, of order N. After the window, the closed vertices in these holes open as in the percolation process restricted to these holes. At time 1 the non-frozen vertices are all open.

Hence the interesting time scale is (5.1.4), moreover it raises the question if there is some kind of limiting process which governs the behaviour of the N-parameter frozen percolation processes as $N \to \infty$ in the scaling window (5.1.4). We have the following, somewhat informal, conjecture:

Conjecture 5.1.8. *When we scale space by N and time according to (5.1.4), we get a non-trivial scaling limit, which is measurable with respect to the near critical ensemble of [45,46]. Moreover, the scaling limit completely describes the frozen clusters of the N-parameter frozen percolation as $N \to \infty$.*

Let us mention some generalizations of our results. We considered the site version of the N-parameter frozen percolation on the triangular lattice above. Straightforward adaptations of the proofs give the same results for the bond version of the N-parameter frozen percolation on the square lattice. See Remark 5.3.7 for more details. Our results remain valid when use some different distance instead of the L^∞ distance in the definition of the N-parameter frozen percolation process, as long as the used distance resembles the L^∞ distance. Examples of such distances include the L^p distances for some $p \geq 1$, or when we rotate the lattice $\mathbb{T}$. Finally let us mention that when we freeze clusters when their *volume* (number of its vertices) reach N, we get a quite different process.

Let us briefly discuss some related results. A version of the N-parameter frozen percolation process on $\mathbb{Z}$ and the binary tree was investigated in [26]. In Section 1.4 we already referred to [4] where Aldous introduced the ∞-parameter frozen percolation process on the binary tree. As we saw in Section 1.4.1, this model has another interesting, so called self organized critical (SOC), behaviour: For all $t > 1/2$, the distribution of the active clusters at time t have the same distribution as critical clusters. Clearly, the N-parameter frozen percolation process on the triangular lattice does not have this property. A mean field version of the frozen percolation model on the complete graph was investigated by Ráth in [80]. He showed that this model has similar SOC properties. Let us mention some results on another closely related model, the so called self-destructive percolation. Van den Berg and Brouwer [91] introduced the model and investigated its properties in the cases where the underlying graph is the binary tree and the square lattice $\mathbb{Z}^2$. Recently, the model on $\mathbb{Z}^d$ for large d [1] and on non-amenable graphs [2] was investigated. Finally, we refer to [16] where a dynamics similar to frozen percolation was investigated on uniform Cayley trees.

The organization of the paper is the following. In Section 5.2, we introduce some more notation, and briefly discuss the results from percolation theory required to prove our main result: We start with some classical correlation inequalities in Section 5.2.1. In Section 5.2.2 we introduce mixed arm events where some of the arms can use only the upper half of the annulus, while others can use the whole annulus. Here we also recall some of their well-known properties and discuss some new ones. In particular, we note that the exponent of the arm events increases when we increase the number of arms which have to stay in the upper half plane. The proof of this statement is postponed to Section 5.A.1 of the Appendix. In Section 5.2.3 we describe the connection between the correlation length with the near critical scaling (5.1.4). We prove Theorem 5.1.3 and Corollary 5.1.7 in Section 5.3 assuming two technical results Proposition 5.3.5 and 5.3.6. In Section 5.4 we introduce some more notation and the notion of thick paths. There we prove Proposition 5.3.6. In this proof a deterministic (combinatorial/geometric) result, Lemma 5.4.5, plays an important role. The proof of this lemma is postponed to Section 5.A.2 of the Appendix. The most technical part of the paper is Section 5.5 where we prove Proposition 5.3.5. In Section 5.5.1 and 5.5.2 we investigate the vertical position of the lowest point of the lowest closed crossing in regions with half open half closed boundary conditions. We combine these results with the ones in Section 5.2 and conclude the proof of Proposition 5.3.5 in Section 5.4. This finishes the proof of the main result.

Acknowledgement

The author thanks Jacob van den Berg, Federico Camia, Pierre Nolin and Gábor Pete for fruitful discussions. He thanks Gábor Pete and Artëm Sapozhnikov for calling his attention to Lemma 5.2.11, which was instrumental in the proof of the main result. He is grateful to Jacob van den Berg, René Conijn and Artëm Sapozhnikov for their numerous comments on the earlier versions of the paper.

5.2 Preliminary results on near critical percolation

We recall some classical results from percolation theory in this section. With suitable modifications, the results of this section also hold for bond percolation on the square lattice unless it is indicated otherwise.

5.2.1 Correlation inequalities

We use the following two inequalities throughout the paper. See Section 2.2 and 2.3 of [50] for more details. We refer to the first theorem as FKG, and as BK for the second.

Definition 5.2.1. *Let $A \subset \{o, c\}^V$ and $U \subseteq V$. We say that an event $A \subset \{o, c\}^V$ is increasing (decreasing) in the configuration in U, if for all $\omega \in A$ we have $\omega' \in A$ where*

$$\omega'(v) = \begin{cases} \omega(v) \text{ or } o\ (c) & \text{for } v \in U \\ \omega(v) & \text{for } v \in V \setminus U. \end{cases}$$

That is, turning some closed (open) vertices in U into open (closed) ones can only help the occurrence of A. In the case where $U = V$ we simply say that A is an increasing (decreasing) event.

Theorem 5.2.2 (FKG)**.** *For any pair of increasing events A, B we have*

$$\mathbb{P}_p (A \cap B) \geq \mathbb{P}_p (A) \mathbb{P}_p (B) .$$

Theorem 5.2.3 (BK)**.** *Let A, B be increasing events, then*

$$\mathbb{P}_p (A \Box B) \leq \mathbb{P}_p (A) \mathbb{P}_p (B) ,$$

where $A \Box B$ denotes the disjoint occurrence of the events A and B.

5.2.2 Mixed arm events, critical arm exponents

Recall the definition of arm events from the introduction. There the arms were allowed to use the whole annulus. We introduce the mixed arm events, where some of the arms lie in the upper half of the annulus, while others can use the whole annulus:

Definition 5.2.4. *Let $l, k \in \mathbb{N}$ with $0 \leq l \leq k$, and a colour sequence σ of length k. Let $v \in V$ and $a, b \in (1, \infty)$ with $a < b$. The full plane k, l mixed arm event with colour sequence σ in the annulus $A(v; a, b)$ is denoted by $\mathcal{A}_{k,l,\sigma}(v; a, b)$. It is the normal k arm event $\mathcal{A}_{k,\sigma}(v; a, b)$ of the Introduction with the extra condition that there is a counter-clockwise ordering of the arms such that the colour of the arms follow σ, and the first l arms lie in the half annulus $A(v; a, b) \cap (\mathbb{Z} \boxtimes [0, \infty) + v)$. When $v = 0$, we omit the first argument from these notations.*

We extend the definition (5.1.3) for mixed arm events by defining

$$\pi_{k,l,\sigma}(a, b) := \mathbb{P}_{1/2} (\mathcal{A}_{k,l,\sigma}(a, b)) .$$

Remark 5.2.5. In the case $k = l$, we get the so called half plane arm events.

We fix $n_0(k) = 10k$ for $k \in \mathbb{N}$. Note that the event $\mathcal{A}_{k,l,\sigma}(n, N)$ is non-empty whenever $n_0(k) < n < N$. Let us summarize the known critical arm exponents for site percolation on the triangular lattice. To our knowledge, Theorem 5.2.6 in its generality is not known to hold for bond percolation on $\mathbb{Z}^2$.

Theorem 5.2.6 (Theorem 3 and 4 of [88])**.** *Let $l, k \in \mathbb{N}$ and σ be a colour sequence of length k. We define $a_{k,l}(\sigma)$*

- *for $k = 1, l = 0$ and any colour sequence σ as*

$$\alpha_{1,0}(\sigma) := \frac{5}{48},$$

- *for $k > 1$ and $l = 0$, when σ contains both colours, as*

$$\alpha_{k,0}(\sigma) := \frac{k^2 - 1}{12},$$

- *for $k = l \geq 1$ and any colour sequence σ as*

$$\alpha_{k,k}(\sigma) := \frac{k(k+1)}{6}.$$

In these cases we have

$$\pi_{k,l,\sigma}(n_0(k), N) = N^{-\alpha_{k,l}(\sigma)+o(1)}$$

as $N \to \infty$,

To our knowledge, for general k and l, neither the value, nor the existence of the exponents is known. We expect that the exponents do exist. We will see in Proposition 5.A.5, that if $\alpha_{k,l}(\sigma)$ and $\alpha_{k,m}(\sigma)$ exists for some $k, l, m \in \mathbb{N}$ and $\sigma \in \{o, c\}^k$ with $m < l$, then $\alpha_{k,m}(\sigma) < \alpha_{k,l}(\sigma)$. Since we do not need such general result, we only prove the following proposition in detail.

Proposition 5.2.7. *For any $k \geq 1$, there are positive constants $c = c(k)$, $\varepsilon = \varepsilon(k)$ such that*

$$\pi_{k,l,\sigma}(n_0(k), N) \leq cN^{-\varepsilon}\pi_{k,0,\sigma}(n_0(k), N) \tag{5.2.1}$$

for $l = 1, 2, \ldots, k$ uniformly in N and in the colour sequence σ.

Remark 5.2.8. (i) We do not need the exact values of the critical exponents of Theorem 5.2.6. For our purposes it is enough to show that certain arm events have exponents at least 2.

(ii) Proposition 5.2.7 and its generalization also hold for mixed arm events in bond percolation on the square lattice.

Proof of Proposition 5.2.7. Proposition 5.2.7 is a simple corollary of Proposition 5.A.3 of the Appendix. Loosely speaking, it states that conditioning on the event that we have k arms in $A(a, b)$, these arms wind around the origin in $O(\log(b/a))$ disjoint sub-annuli of $A(a, b)$ with probability at least $1 - \left(\frac{a}{b}\right)^{\kappa}$ for some $\kappa > 0$. The proof of Proposition 5.A.3 can be found in the Appendix. □

Remark 5.2.9. Recall that we do not know in general if the exponents $\alpha_{k,l}(\sigma)$ exist or not. Nonetheless, on the triangular lattice, Proposition 5.2.7 and Theorem 5.2.6 and the BK inequality (Theorem 5.2.3) give that for any colour sequence σ, there is an upper bound with exponent strictly larger than 2 for $\pi_{k,l,\sigma}(n_0(k), N)$ when

- $k \geq 6$, and $l \geq 0$, or
- $k \geq 5$ and $l \geq 1$, or
- $k \geq 4$ and $l \geq 3$.

For arm events with exponents larger than 2 in the case of bond percolation on the square lattice see Remark 5.2.14 below.

Another well-known attribute of critical arm events is their quasi-multiplicative property. For the full plane, respectively for half plane, arm events this property is shown to hold in Proposition 17 of [77], respectively in Section 1.4.6 of [77]. Simple modifications of these arguments apply to mixed arm events. We introduce the notation $\asymp$ when the ratio of the two quantities is bounded away from 0 and ∞. We have:

Proposition 5.2.10. *Let $k \geq 1$ and $\sigma \in \{o, c\}^k$. Then*

$$\pi_{k,l,\sigma}(n_1, n_2)\, \pi_{k,l,\sigma}(n_2, n_3) \asymp \pi_{k,l,\sigma}(n_1, n_3)$$

uniformly in $n_0(k) \leq n_1 \leq n_2 \leq n_3$.

In the following lemma we consider arm events where the open arms are p-open and the closed arms are q-closed where $p, q \in [0, 1]$ with p not necessarily equal to q. When p and q are of the form (5.1.4), then we call these arm events near critical arm events. In this case the probabilities of these events are comparable to critical arm event probabilities. The following lemma is a generalization of Lemma 2.1 of [44] and Lemma 6.3 of [34].

Lemma 5.2.11. *Let $v \in V$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $a, b \in (0, 1)$ with $a < b$. Let $\mathcal{A}^{\lambda_1,\lambda_2,N}_{k,l,\sigma}(v; aN, bN)$ denote the modification of the event $\mathcal{A}_{k,l,\sigma}(v; aN, bN)$ where the open arms are $p_{\lambda_2}(N)$-open and the closed arms are $p_{\lambda_1}(N)$-closed. Then there are positive constants $c = c(\lambda_1, \lambda_2, k)$ and $N_0 = N_0(\lambda_1, \lambda_2, a, b, k)$ such that*

$$\mathbb{P}\left(\mathcal{A}^{\lambda_1,\lambda_2,N}_{k,l,\sigma}(v; aN, bN)\right) \leq c\pi_{k,l,\sigma}(aN, bN)$$

for $N \geq N_0$.

Proof of Lemma 5.2.11. It follows from either of the proof of Lemma 2.1 of [44] or from the proof of Lemma 6.3 of [34]. □

In the following events we collect some of the near critical arm events which have upper bounds with exponents strictly larger than 2. These events play a crucial role in our main result.

Definition 5.2.12. *Let $a, b \in (0, 1)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, $K > 0$ and $N \in \mathbb{N}$ with $a < b$. Let $\mathcal{NA}^c(a, b, \lambda_1, \lambda_2, K, N)$ denote the union of the events $\mathcal{A}^{\lambda_1,\lambda_2,N}_{k,l,\sigma}(v; aN, bN)$ for $(k, l) \in \{(4, 3), (5, 1), (6, 0)\}$, $\sigma \in \{o, c\}^k$, $v \in B(KN)$ as well as the versions of these events where the half plane arms can only use the lower, left or right half of the annulus $A(v; aN, bN)$. We define $\mathcal{NA}(a, b, \lambda_1, \lambda_2, K, N)$ as the complement of the event above.*

We show that for fixed b, K, λ_1 and λ_2, we can set $a \in (0,1)$ so that the probability of $\mathcal{NA}(a, b, \lambda_1, \lambda_2, K, N)$ becomes as close to 1 as we require for large N. More precisely, we prove the following:

Corollary 5.2.13. *There is $\tilde{\varepsilon} > 0$ such that for all $a, b \in (0,1)$, with $a < b$ and $\lambda_1, \lambda_2 \in \mathbb{R}$ there are positive constants $c = c(\lambda_1, \lambda_2, K)$ and $N_0 = N_0(a, b, \lambda_1, \lambda_2, K)$ such that*

$$\mathbb{P}(\mathcal{NA}(a, b, \lambda_1, \lambda_2, K, N)) \geq 1 - c\frac{a^{\tilde{\varepsilon}}}{b^{2+\tilde{\varepsilon}}}$$

for $N \geq N_0$.

Proof of Corollary 5.2.13. Suppose that one of the arm events in Definition 5.2.12, for example $\mathcal{A}_{k,l,\sigma}^{\lambda_1,\lambda_2,N}(v; aN, bN)$ for some $v \in B(KN)$, occurs. Then the event $\mathcal{A}_{k,l,\sigma}^{\lambda_1,\lambda_2,N}\left(\lfloor 2aN \rfloor z; 2aN, \frac{b}{2}N\right)$ occurs for some $z \in V \cap B\left(\left\lceil \frac{a+K}{2a} \right\rceil\right)$.

Combination of Remark 5.2.9 and Lemma 5.2.11 gives that there are constants $c' = c'(\lambda_1, \lambda_2)$ and $N_0 = N_0(a, b, \lambda_1, \lambda_2)$, and a universal constant $\tilde{\varepsilon} > 0$ such that the probability of one of these events is at most

$$c'\left(\frac{2a}{b/2}\right)^{2+\varepsilon} \tag{5.2.2}$$

for $N \geq N_0$. The same argument works for other arm events which appear in Definition 5.2.12, and provide an upper bound similar to (5.2.2). Hence (5.2.2) combined with $\left|B\left(\left\lceil \frac{a+K}{2a} \right\rceil\right)\right| = O\left(a^{-2}\right)$ concludes the proof of Corollary 5.2.13. $\square$

Remark 5.2.14. To our knowledge it is not known if the direct analogue of Corollary 5.2.13 holds on the square lattice. The reason is that the exponent $\alpha_{5,0}(\sigma)$ and $\alpha_{3,3}(\sigma)$ is not known for general σ. See Remark 26 of [77].

We recall the proof of Theorem 24 and Remark 26 of [77], where it is shown that $\alpha_{5,0}(o, c, o, o, c) = 2$ and $\alpha_{3,3}(c, o, c) = 2$ on the square lattice. This implies that a version of Corollary 5.2.13 holds for the square lattice if we modify Definition 5.2.12 so that we only forbid the occurrence of those arm events where the required set of arms contain

- three half plane arm events with colour sequence (o, c, o) or (c, o, c), or
- five full plane arms with colour sequence (o, c, o, o, c) or (c, o, c, c, o)

as a subset.

5.2.3 Near-critical scaling and correlation length

Recall that in Section 5.1 we already gave an explanation for the near critical parameter scale (5.1.4). In this section we give a different interpretation of this parameter scale, which is connected to the correlation length introduced by Kesten in [68].

We say that there is an open (closed) horizontal crossing of a parallelogram $B := [a,b] \boxtimes [c,d]$ if there is an open (closed) path connecting $\{\lceil a \rceil\} \boxtimes [c,d]$ and $\{\lfloor b \rfloor\} \boxtimes [c,d]$ in$[a,b] \boxtimes [c,d]$. For the event that there is an open (closed) horizontal crossing of B we use the notation $\mathcal{H}_o(B)$ ($\mathcal{H}_c(B)$). One can define similar events for vertical crossings, which we denote by $\mathcal{V}_o(B)$ and $\mathcal{V}_c(B)$. For $\varepsilon \in (0,1/2)$ the correlation length is defined as

$$L_\varepsilon(p) = \begin{cases} \min\{n \mid \mathbb{P}_p(\mathcal{H}_o(B(n))) \le \varepsilon\} & \text{when } p < p_c \\ \min\{n \mid \mathbb{P}_p(\mathcal{H}_o(B(n))) \ge 1-\varepsilon\} & \text{when } p > p_c. \end{cases}$$

Remark 5.2.15. The particular choice of ε is not important in this definition. Indeed, Corollary 37 of [77], or alternatively Corollary 2 of [68], gives that

$$L_\varepsilon(p) \asymp L_{\varepsilon'}(p)$$

for any $\varepsilon, \varepsilon' \in (0,1/2)$ uniformly in $p \in (0,1)$.

We show that the control over the near critical parameter λ gives a control over the correlation length in Corollary 5.2.17 and 5.2.18 below. Recall the remark after Lemma 8 of [68]:

Proposition 5.2.16. *For any fixed $\varepsilon \in (0,1/2)$, we have*

$$|p-p_c|\,(L_\varepsilon(p))^2\,\pi_{4,0,alt}(1, L_\varepsilon(p)) \asymp 1$$

uniformly for $p \neq 1/2$.

Note that for fixed $\varepsilon > 0$, the correlation length $L_\varepsilon(p)$ is a decreasing (increasing) function of p for $p > p_c$ ($p < p_c$). Combination of this and Proposition 5.2.10 we get:

Corollary 5.2.17. *For all $\lambda \in \mathbb{R} \setminus \{0\}$ and $\varepsilon \in (0,1/2)$,*

$$L_\varepsilon(p_\lambda(N)) \asymp N. \tag{5.2.3}$$

Corollary 5.2.18. *For any $C > 0$ and $\varepsilon \in (0,1/2)$ there exits $\lambda_1 = \lambda_1(C,\varepsilon) > 0$ and $N_1 = N_1(C,\varepsilon)$ such that for any $\lambda \in \mathbb{R}$ with $|\lambda| \ge \lambda_1$ we have*

$$L_\varepsilon(p_\lambda(N)) \le CN$$

for $N \ge N_1$. Also, for any $c > 0$, and $\varepsilon \in (0,1/2)$ there exists $\lambda_2(c,\varepsilon) > 0$ and $N_2 = N_2(c,\varepsilon)$ such that for any $\lambda \in \mathbb{R} \setminus \{0\}$ with $|\lambda| \le \lambda_2$ we have

$$L_\varepsilon(p_\lambda(N)) \ge cN$$

for $N \ge N_2$.

Remark 5.2.19. On the triangular lattice, a ratio limit theorem for $\pi_{4,0,alt}$, Proposition 4.7 of [46] holds. This combined with the definition of $L_\varepsilon(p)$, and Proposition 5.2.16 shows that the following stronger statement holds on the triangular lattice:

Claim. For all $\lambda_1, \lambda_2 \in \mathbb{R}$ with $\lambda_1 \leq \lambda_2$, $\lambda_1\lambda_2 > 0$ and $\varepsilon \in (0, 1/2)$ there are positive constants $c = c(\varepsilon), C = C(\varepsilon)$ and $N_0 = N_0(\varepsilon, \lambda_1, \lambda_2)$ such that

$$cN|\lambda|^{-4/3} \leq L_\varepsilon(p_\lambda(N)) \leq CN|\lambda|^{-4/3}$$

for all $\lambda \in [\lambda_1, \lambda_2]$ and $N \geq N_0$.

Standard Russo-Seymour-Welsh (RSW) techniques and the definition of the correlation length give that the control over the correlation length gives a control over the crossing probabilities of parallelograms. This combined with the two corollaries above show that the control over the near critical parameter gives control over the crossing probabilities. See Corollary 5.2.20 and 5.2.21 below:

Corollary 5.2.20. *For all $\lambda \in \mathbb{R}$ and $a, b \in (0, \infty)$, there are constants $c = c(a, b, \lambda) \in (0, 1)$, $C = C(a, b, \lambda) \in (0, 1)$ and $N_0 = N_0(a, b, \lambda)$ such that*

$$c < \mathbb{P}_{p_\lambda(N)}(\mathcal{H}_o([0, aN] \boxtimes [0, bN])) < C$$
$$c < \mathbb{P}_{p_\lambda(N)}(\mathcal{H}_c([0, aN] \boxtimes [0, bN])) < C$$

for $N \geq N_0$.

Corollary 5.2.21. *Let $\delta \in (0, 1)$, and $a, b \in (0, \infty)$. There exists $\lambda_1 = \lambda_1(\delta, a, b) > 0$ and $N_1 = N_1(\delta, a, b)$ such that for all $\lambda \geq \lambda_1$*

$$\mathbb{P}_{p_\lambda(N)}(\mathcal{H}_o([0, aN] \boxtimes [0, bN])) > 1 - \delta$$

for $N \geq N_1$. Furthermore, there exists $\lambda_2 = \lambda_2(\delta, a, b) < 0$ and $N_2 = N_2(\delta, a, b)$ such that for all $\lambda \leq \lambda_2$

$$\mathbb{P}_{p_\lambda(N)}(\mathcal{H}_c([0, aN] \boxtimes [0, bN])) > 1 - \delta$$

for $N \geq N_2$.

Similar RSW techniques show that it is unlikely to have crossing in a thin and long parallelogram in the hard direction in the critical window. See Remark 40 [77] for more details.

Corollary 5.2.22. *Let $\lambda \in \mathbb{R}$, and $a, b \in (0, 1)$. There exists positive constants $c = c(\lambda), C = C(\lambda)$ and $N_0 = N_0(\lambda, a, b)$ such that*

$$\mathbb{P}_{p_\lambda(N)}(\mathcal{H}_o([0, aN] \boxtimes [0, bN])) \leq C \exp\left(-c\frac{a}{b}\right)$$

for $N \geq N_0$.

The following event plays a crucial role in the proof of our main result.

Definition 5.2.23. *Let $a, b \in (0, 1)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $N \in \mathbb{N}$ with $a < b$. Let $\mathcal{NC}(a, b, \lambda_1, \lambda_2, K, N)$ denote the event that for all parallelograms $B = [0, aN] \boxtimes [0, bN] + z$ with $z \in B(KN)$, there is neither a $p_{\lambda_1}(N)$-open nor a $p_{\lambda_2}(N)$-closed horizontal crossing in B.*

The following Corollary 5.2.24 follows from Corollary 5.2.22 by arguments analogous to the proof of Corollary 5.2.13.

Corollary 5.2.24. *Let $a, b \in (0,1)$, $\lambda_1, \lambda_2 \in \mathbb{R}$, and $N \in \mathbb{N}$ with $a < b$. There are positive constants $c = c(\lambda_1, \lambda_2)$, $C = C(\lambda_1, \lambda_2)$ and $N_0 = N_0(a, b, \lambda_1, \lambda_2)$ such that*

$$\mathbb{P}\left(\mathcal{NC}\left(a, b, \lambda_1, \lambda_2, K, N\right)\right) \geq 1 - Ca^{-2} \exp\left(-c\frac{b}{a}\right)$$

for $N \geq N_0$.

We finish this section by stating two lemmas which will be used explicitly in the proof of our main result.

Lemma 5.2.25. *For any fixed $\lambda \in \mathbb{R}$, for any $a, b \in (0, \infty)$ and $\varepsilon > 0$, there is are positive integer $k = k(\lambda, a, b, \varepsilon)$ and $N_0 = N_0(\lambda, a, b, \varepsilon)$ such that*

$$\mathbb{P}_{p_\lambda(N)}\left(\textit{there are at least } k \textit{ disjoint closed arms in } A\left(aN, bN\right)\right) < \varepsilon$$

for $N \geq N_0$.

Proof of Lemma 5.2.25. This is a consequence of Corollary 5.2.20 and the BK inequality (Theorem 5.2.3). The proof also appears in the proof of Lemma 15 of [77]. □

Definition 5.2.26. *Let $a, b, c, d, f \in \mathbb{R}$ with $a \leq b$, $c \leq d$ and $f > 0$. We say that there is an open (closed) f-net in $B = [a, b] \boxtimes [c, d]$ if there is an open (closed) vertical crossing in the parallelograms $[a + i\lfloor f \rfloor, a + (i+1)\lfloor f \rfloor - 1] \boxtimes [c, d]$, and there is an open (closed) horizontal crossing in the parallelograms $[a, b] \boxtimes [c + j\lfloor f \rfloor, c + (j+1)\lfloor f \rfloor - 1]$ for $i = 0, 1, \ldots, \lfloor (b-a)/\lfloor f \rfloor \rfloor$ and $j = 0, 1, \ldots, \lfloor (d-c)/\lfloor f \rfloor \rfloor$.*

For $\lambda \in \mathbb{R}$ and $\delta \in (0, \infty)$, $\mathcal{N}_c(\lambda, \delta, K, N)$ ($\mathcal{N}_o(\lambda, \delta, K, N)$) denotes the event that there is a $p_\lambda(N)$-closed ($p_\lambda(N)$-open) δN-net in $B(KN)$.

Lemma 5.2.27. *Let $\varepsilon, \delta, K > 0$. There exists $\lambda_1 = \lambda_1(\varepsilon, \delta, K) \in \mathbb{R}$ and $N_1 = N_1(\varepsilon, \delta, K)$ such that*

$$\mathbb{P}\left(\mathcal{N}_o\left(\lambda_1, \delta, K, N\right)\right) > 1 - \varepsilon$$

for $N \geq N_1$. Moreover there exists $\lambda_2 = \lambda_2(\varepsilon, \delta, K) \in \mathbb{R}$ and $N_2 = N_2(\varepsilon, \delta, K)$ such that

$$\mathbb{P}\left(\mathcal{N}_c\left(\lambda_2, \delta, K, N\right)\right) > 1 - \varepsilon$$

for $N \geq N_2$.

Proof of Lemma 5.2.27. This is a consequence of Corollary 5.2.21 and the FKG inequality (Theorem 5.2.2). □

5.3 Proof of the main results

We prove our main results Theorem 5.1.3 and Corollary 5.1.7 in this section assuming Proposition 5.3.5 and 5.3.6.

Definition 5.3.1. *In the N-parameter frozen percolation process we call a vertex* frozen *at some time $t \in [0,1]$, if either it or one of its neighbours have an open cluster with diameter bigger than N at time t. If a site is not frozen at time t, then we say it is* active *at time t. Note that both frozen and active sites can be open or closed. We say that F is a* (open) frozen cluster *at time $t \in [0,1]$ if it is a connected component of the open vertices at time t with* $\operatorname{diam}(F) \geq N$. *In the case where $t = 1$, we simply say that F is a frozen cluster.*

Recall Definition 5.2.26. We observe the following.

Observation 5.3.2. *Let $K > 0$ and $N \in \mathbb{N}$. Then in the N-parameter frozen percolation process there is no frozen cluster at time $p_\lambda(N)$ in $B(KN)$ on the event $\mathcal{N}_c(\lambda, 1/6, K+2, N)$. Hence on $\mathcal{N}_c(\lambda, 1/6, K+2, N)$, a vertex in $B(KN)$ is open (closed) in the N-parameter frozen percolation process at time $p_\lambda(N)$ if and only if it is $p_\lambda(N)$-open ($p_\lambda(N)$-closed).*

We show that the number of frozen clusters intersecting $B(KN)$ in the N-parameter frozen percolation process is tight in N.

Lemma 5.3.3. *Let $K > 0$ and $N \in \mathbb{N}$. Let $FC(t,K) = FC(t,K,N)$ denote the number of frozen clusters intersecting $B(KN)$ at time $t \in [0,1]$ in the N-parameter frozen percolation process. Then for all $\varepsilon > 0$ there exists $L = L(\varepsilon, K)$ and $N_0 = N_0(\varepsilon, K)$ such that*

$$\mathbb{P}_N(FC(1,K) > L) < \varepsilon$$

for $N \geq N_0$.

Proof of Lemma 5.3.3. By Lemma 5.2.27 we set $\lambda = \lambda(\varepsilon, K) \in \mathbb{R}$ such that

$$\mathbb{P}_N(\mathcal{N}_c(\lambda, 1/6, K+4, N)) > 1 - \frac{1}{2}\varepsilon \tag{5.3.1}$$

for $N \geq N_1(\varepsilon, K)$. Let F be an open frozen cluster which intersects $B(KN)$. From Observation 5.3.2 we get the vertices of ∂F are closed at $p_\lambda(N)$ in the N-parameter percolation process on the event $\mathcal{N}_c(\lambda, 1/6, K+4, N)$.

Let us cover the parallelogram $B(KN)$ with the annuli

$$A_z = A(\lfloor N/20 \rfloor z; \lfloor N/20 \rfloor, \lfloor N/10 \rfloor) \text{ with } z \in B(\lceil 20K \rceil).$$

Suppose that there is an open frozen cluster in the N-parameter frozen percolation which has a vertex in $B(KN)$. The construction of the annuli above gives that there is $z \in B(\lceil 20K \rceil)$ such that $B(\lfloor N/20 \rfloor z; \lfloor N/20 \rfloor)$, the inner parallelogram of A_z, contains a vertex of this open frozen cluster. Since the diameter of $B(\lfloor N/20 \rfloor z; \lfloor N/10 \rfloor)$ is less than N, this cluster has to cross the annulus A_z. Hence for each open frozen cluster intersecting $B(KN)$, we find at least one open frozen crossing of an annulus A_z. Moreover, if there are $k \geq 2$ different frozen clusters crossing the annulus A_z, then there are at least k disjoint closed frozen arms which separate the open frozen clusters in A_z at time 1. By the arguments above, these arms are $p_\lambda(N)$-closed. Thus the number of different frozen clusters intersecting $B(\lfloor N/20 \rfloor z; \lfloor N/20 \rfloor)$ is bounded above by $1 \vee l_z$, where l_z is the number of disjoint $p_\lambda(N)$-closed arms of A_z. Hence by the translation variance of the N-parameter frozen percolation process we have

$$\begin{aligned}
&\mathbb{P}_N\left(FC(1,K) \geq L, \mathcal{N}_c(\lambda, 1/24)\right) \\
&\qquad \leq \mathbb{P}_{p_\lambda(N)}\left(\sum_{z \in B(\lceil 20K \rceil)} (1 \vee l_z) \geq L\right) \\
&\qquad \leq \mathbb{P}_{p_\lambda(N)}\left(\exists z \in B(\lceil 20K \rceil) \text{ such that } l_z \geq (2\lceil 20K \rceil + 1)^{-2} L\right) \\
&\qquad \leq (2\lceil 20K \rceil + 1)^2 \, \mathbb{P}_{p_\lambda(N)}\left(l_0 \geq (2\lceil 20K \rceil + 1)^{-2} L\right) \qquad (5.3.2)
\end{aligned}$$

By Lemma 5.2.25 we set $L = L(\varepsilon, K) \geq (2\lceil 20K \rceil + 1)^2$ and $N_2 = N_2(\varepsilon, K)$ such that

$$\mathbb{P}_{p_\lambda(N)}\left(l_0 \geq L/200^2\right) < \frac{1}{2}(2\lceil 20K \rceil + 1)^{-2} \varepsilon$$

for $N \geq N_2$. This combined with (5.3.2) gives that

$$\mathbb{P}_N\left(FC(1,K) \geq L, \, \mathcal{N}_c(\lambda, 1/6, K+4, N)\right) < \frac{1}{2}\varepsilon \qquad (5.3.3)$$

for $N \geq N_2$. We set $N_0 := N_1 \vee N_2$. A combination of (5.3.1) and (5.3.3) finishes the proof of Lemma 5.3.3. $\square$

Definition 5.3.4. *For $v \in V$ and $\lambda \in \mathbb{R}$ let $\mathcal{C}_a(v;\lambda) = \mathcal{C}_a(v;\lambda,N)$ denote the active cluster of v in the N-parameter frozen percolation process at time $p_\lambda(N)$. We omit the first argument from the notation above when $v = \underline{0}$.*

We state the two propositions below which play a crucial role in the proof of Theorem 5.1.3. The proof of these propositions are rather technical, so we postpone them to the next section. The first proposition shows that for $\alpha > 0$, it is unlikely to have an active cluster at time $p_\lambda(N)$ which intersects $B(KN)$ and has diameter close to αN.

Proposition 5.3.5. *For all $\lambda \in \mathbb{R}$ and $\varepsilon, K, \alpha > 0$, there exist $\theta = \theta(\lambda, \alpha, \varepsilon, K) \in (0, 1/2)$ and $N_0 = N_0(\lambda, \alpha, \varepsilon, K)$ such that*

$$\mathbb{P}_N\left(\exists v \in B(KN) \textit{ s.t. } \operatorname{diam}(\mathcal{C}_a(v;\lambda)) \in ((\alpha-\theta)N, (\alpha+\theta)N)\right) < \varepsilon$$

for $N \geq N_0$.

The second proposition claims that if there is a vertex $v \in B(KN)$ such that $\operatorname{diam}(\mathcal{C}_a(v;\lambda_1,N)) \geq (1+\theta)N$ then some part of $\mathcal{C}_a(v;\lambda_1,N)$ freezes ‘soon’:

Proposition 5.3.6. *Let $\theta \in (0,1)$, $\varepsilon > 0$ and $\lambda_1, K, \in \mathbb{R}$. Recall the notation $FC(t, K+2, N)$ from Lemma 5.3.3. There exists $\lambda_2 = \lambda_2(\lambda_1, \theta, \varepsilon)$ and $N_0 = N_0(\lambda_1, \theta, \varepsilon)$ such that the probability of the intersection of the events*

- *$\exists v \in B(KN)$ such that* $\operatorname{diam}(\mathcal{C}_a(v; \lambda_1, N)) \geq (1+\theta) N$, *and*
- *none of the clusters intersecting $B((K+2)N)$ freeze in the time interval $(p_{\lambda_1}(N), p_{\lambda_2}(N)]$, i.e.*

$$FC(p_{\lambda_1}(N), K+2, N) = FC(p_{\lambda_2}(N), K+2, N)$$

is less than ε for $N \geq N_0$.

Before we turn to the proof of our main results we make a remark on how to adapt the proofs for the N-parameter frozen bond percolation process on the square lattice.

Remark 5.3.7. The arguments in Section 5.3, 5.4, 5.5 and in the Appendix can be easily adapted to the N-parameter frozen bond percolation on the square lattice. Some care is required when we use Corollary 5.2.13: As we already noted in Remark 5.2.14, the direct analogue of Corollary 5.2.13 does not hold on the square lattice. However, one can check that the version of Corollary 5.2.13 which was proposed in Remark 5.2.14 is enough for the proofs appearing in Section 5.3, 5.4, 5.5.

5.3.1 Proof of Theorem 5.1.3

Proof of Theorem 5.1.3. The proof follows the following informal strategy. Consider the following procedure. We set $\lambda_1 = 0$. We look at the N-parameter percolation process at time $p_{\lambda_1}(N)$. We have two cases.

In the first case all the active clusters at time $p_{\lambda_1}(N)$ intersecting $B(KN)$ have diameter less than N. Hence no cluster intersecting $B(KN)$ can freeze after $p_{\lambda_1}(N)$. We terminate the procedure.

In the second case there is $v \in B(KN)$ such that $\operatorname{diam}(\mathcal{C}_a(v; \lambda_1, N)) \geq N$. Using Proposition 5.3.5 we set θ_1 such that the diameter of this cluster is at least $(1+\theta_1) N$ with probability close to 1. If $\operatorname{diam}(\mathcal{C}_a(v; \lambda_1, N)) \leq (1+\theta_1) N$, then we stop the procedure. In the case where $\operatorname{diam}(\mathcal{C}_a(v; \lambda_1, N)) > (1+\theta_1) N$, then using Proposition 5.3.6 we set $\lambda_2 \geq \lambda_1$ such that some part of $\mathcal{C}_a(v; \lambda_1, N) \cap B((K+2)N)$ freezes in the time interval $[p_{\lambda_1}(N), p_{\lambda_2}(N)]$ with probability close to 1. If indeed some part of $\mathcal{C}_a(v; \lambda_1, N) \cap B((K+2)N)$ freezes in the time interval $[p_{\lambda_1}(N), p_{\lambda_2}(N)]$, then we iterate the procedure starting from time $p_{\lambda_2}(N)$. Otherwise we terminate the procedure.

Using Lemma 5.3.3 we set L such that the event where there are at least L frozen clusters intersecting $B((K+2)N)$ at time 1 has probability smaller than $\varepsilon/2$. In each step of the procedure either the procedure stops, or the number of frozen clusters intersecting $B((K+2)N)$ increases by at least 1. Hence the event that the procedure runs for at least L steps has probability at most $\varepsilon/2$.

Moreover, we set the parameters λ_i, θ_i for $i \geq 1$ above such that with probability at least $1 - \varepsilon/2$ we terminate the procedure when there are no active clusters intersecting $B(KN)$ with diameter at least N. Thus with probability at least $1-\varepsilon$ the procedure stops within L steps, and we stop when there are no active clusters with diameter at least N intersecting $B(KN)$. Hence $\lambda = \lambda_{L+1}$ satisfies the conditions of Theorem 5.1.3, which finishes the proof of Theorem 5.1.3.

Let us turn to the precise proof. Recall the notation $FC(t, K)$ from Lemma 5.3.3. The same lemma gives that there is $L = L(\varepsilon, K)$ and $N_1' = N_1'(\varepsilon, K)$ such that

$$\mathbb{P}_N\left(FC(1, K+2) \geq L\right) \leq 2^{-1}\varepsilon. \tag{5.3.4}$$

We define the deterministic sequence $(\lambda_i, N_i', \theta_i, N_i'')_{i\in\mathbb{N}}$ inductively as follows. We start by setting $\lambda_1 = 0$.

Suppose that we have already defined λ_i for some $i \in \mathbb{N}$. We use Proposition 5.3.5 to set $\theta_i = \theta_i(\varepsilon)$ and $N_i'' = N_i''(\varepsilon)$ such that

$$\mathbb{P}_N\left(\exists v \in B(KN) \text{ s.t. } \operatorname{diam}\left(\mathcal{C}_a(v, \lambda_i)\right) \in [N, (1+\theta_i)N)\right) < \varepsilon 2^{-i-2}$$

for $N \geq N_i''$.

Suppose that we have already defined θ_i for some $i \in \mathbb{N}$. Then by Proposition 5.3.6 we set $\lambda_{i+1} = \lambda_{i+1}(\varepsilon)$ and $N_{i+1}' = N_{i+1}'(\varepsilon)$ such that the probability of the intersection of the events

- $\exists v \in B(KN)$ such that $\operatorname{diam}\left(\mathcal{C}_a(v; \lambda_i)\right) \geq (1+\theta_i)N$, and
- $FC\left(p_{\lambda_i}(N), K+2\right) = FC\left(p_{\lambda_{i+1}}(N), K+2\right)$

is less than $2^{-i-2}\varepsilon$ for $N \geq N_{i+1}'$. Note that the event

$$\left\{FC\left(p_{\lambda_i}(N), K+2\right) = FC\left(p_{\lambda_{i+1}}(N), K+2\right),\ FC\left(p_{\lambda_i}(N), K\right) < FC(1, K)\right\}$$

is a subset of the union of the events appearing in the definition of θ_i and λ_{i+1} for $i \geq 1$. Thus the construction above gives that

$$\mathbb{P}_N\begin{pmatrix} FC\left(p_{\lambda_i}(N), K+2\right) = FC\left(p_{\lambda_{i+1}}(N), K+2\right), \\ FC\left(p_{\lambda_i}(N), K\right) < FC(1, K) \end{pmatrix} \leq 2^{-i-1}\varepsilon \tag{5.3.5}$$

for $i \geq 1$.

We set $N_0 = \bigvee_{i=1}^{L+1}\left(N_i' \vee N_i''\right)$. By (5.3.4) we have

$$\begin{aligned}
&\mathbb{P}_N\big(\text{a cluster intersecting } B(KN) \text{ freezes after time } p_{\lambda_{L+1}}(N)\big) \\
&\quad = \mathbb{P}_N\left(FC\left(p_{\lambda_{L+1}}(N), K\right) < FC(1, K)\right) \\
&\quad \leq \mathbb{P}_N\left(L < FC(1, K+2)\right) \\
&\qquad + \mathbb{P}_N\left(FC\left(p_{\lambda_{L+1}}(N), K+2\right) \leq L,\ FC\left(p_{\lambda_{L+1}}(N), K\right) < FC(1, K)\right) \\
&\quad \leq \varepsilon/2 + \mathbb{P}_N\left(\bigcup_{i=1}^{L+1}\left\{\begin{matrix} FC\left(p_{\lambda_i}(N), K+2\right) = FC\left(p_{\lambda_{i+1}}(N), K+2\right), \\ FC\left(p_{\lambda_{i+1}}(N), K\right) < FC(1, K) \end{matrix}\right\}\right)
\end{aligned}$$

$$\leq \varepsilon/2 + \sum_{i=1}^{L+1} \mathbb{P}_N \begin{pmatrix} FC\left(p_{\lambda_i}(N), K+2\right) = FC\left(p_{\lambda_{i+1}}(N), K+2\right), \\ FC\left(p_{\lambda_{i+1}}(N), K\right) < FC(1, K) \end{pmatrix}$$
$$\leq \varepsilon/2 + \sum_{i=1}^{L+1} 2^{-i-1}\varepsilon < \varepsilon$$

for $N \geq N_0$ where we applied (5.3.5) in the last line. This finishes the proof of Theorem 5.1.3. $\square$

5.3.2 Proof of Corollary 5.1.7

Proof of Corollary 5.1.7 . For $\lambda \in \mathbb{R}$ and $N \in \mathbb{N}$ let $NF(\lambda) = NF(\lambda, N)$ denote the event that no cluster intersecting $B(5N)$ freezes after time $p_\lambda(N)$. By Theorem 5.1.3 there is $\lambda = \lambda(\varepsilon)$ and $N_1 = N_1(\varepsilon)$ such that

$$\mathbb{P}_N(NF(\lambda)) > 1 - \varepsilon/3 \tag{5.3.6}$$

for $N \geq N_1$.

First we consider the case where the origin is in an open frozen cluster at time 1, that is $\operatorname{diam}(C(1)) \geq N$. Note that on the event $NF(\lambda)$, this frozen cluster was formed before or at $p_\lambda(N)$. Hence on this event there is a $p_\lambda(N)$-open path from the origin to distance at least $N/2$. Hence the event $\mathcal{A}_{1,0,o}^{\lambda,\lambda,N}(1, N/2)$ defined in Lemma 5.2.11 occurs.

Let us turn to the case where $\operatorname{diam}(C(1)) < N$. Recall the notation $\mathcal{C}_a(\lambda)$ from Definition 5.3.4. It is easy to check that $C(1) = \mathcal{C}_a(\lambda)$ on the event $\{\operatorname{diam}(C(1)) < N\} \cap NF(\lambda)$.

If $\operatorname{diam}(\mathcal{C}_a(\lambda)) < aN$, then $\partial\mathcal{C}_a(\lambda) \cap B(2aN) \neq \emptyset$ for large N. Since $v \in \partial\mathcal{C}_a(\lambda) \cap B(2aN)$ is frozen, it has a neighbour which has an open frozen path to distance at least $N/2$. On the event $NF(\lambda)$, this path is $p_\lambda(N)$-open. Hence the event $\mathcal{A}_{1,0,o}^{\lambda,\lambda,N}(2aN, N/2)$ occurs. This combined with the argument above, for $a \in (0,1)$ and $N > N_2 = 1/a$ we have

$$\{\operatorname{diam}(C(1)) \in [0, aN) \cup [N, \infty)\} \cap NF(\lambda) \subseteq \mathcal{A}_{1,0,o}^{\lambda,\lambda,N}(2aN, N/2).$$

Hence by Lemma 5.2.11 there is $c = c(\lambda)$ and $N_3 = N_3(\lambda)$ such that

$$\begin{aligned}\mathbb{P}_N\left(\operatorname{diam}(C(1)) \in [0, aN) \cup [N, \infty), NF(\lambda)\right) &\leq \mathbb{P}\left(\mathcal{A}_{1,0,o}^{\lambda,\lambda,N}(2aN, N/2)\right) \\ &\leq c\mathbb{P}_{1/2}\left(\mathcal{A}_{1,o}(2aN, N/2)\right)\end{aligned}$$

for $N \geq N_3$. Theorem 5.2.6 gives that there is $a = a(\varepsilon)$ and $N_4 = N_4(\varepsilon)$ such that

$$\mathbb{P}_N\left(\operatorname{diam}(C(1)) \in [0, aN) \cup [N, \infty), NF(\lambda)\right) \leq c\mathbb{P}_{1/2}\left(\mathcal{A}_{1,o}(2aN, N/2)\right) < \varepsilon/3. \tag{5.3.7}$$

for $N \geq N_4$.

Finally, Proposition 5.3.5 gives $b = b(\varepsilon)$ and $N_5 = N_5(\varepsilon)$ such that

$$\begin{aligned}\mathbb{P}_N\left(\mathrm{diam}\left(\mathcal{C}_a(\lambda)\right) \in [bN, N),\ NF(\lambda)\right) &\leq \mathbb{P}_N\left(\mathrm{diam}\left(\mathcal{C}_a(\lambda)\right) \in [bN, N)\right) \\ &\leq \varepsilon/3 \end{aligned} \tag{5.3.8}$$

for $N \geq N_5$.

Since $C(1) = \mathcal{C}_a(\lambda)$ on the event $\{\mathrm{diam}(C(1)) < N\} \cap NF(\lambda)$, a combination of (5.3.6), (5.3.7) and (5.3.8) finishes the proof of Corollary 5.1.7. □

5.4 Proof of Proposition 5.3.6

5.4.1 Notation

Let us introduce some more notation. For $u = (u_1, u_2), v = (v_1, v_2) \in V$, we say that u is left (right) of v if $u_1 \leq v_1$ ($u_1 \geq v_1$). Similarly we say that u is below (above) v if $u_2 \leq v_2$ ($u_2 \geq v_2$). For a finite set of vertices $W \subseteq V$ we say that $v = (v_1, v_2) \in W$ is a leftmost (rightmost) vertex of W if for all $w = (w_1, w_2) \in W$, $v_1 \leq w_1$ ($v_1 \geq w_1$). We define the lowest and highest vertices of W in an analogous way.

Recall that $v, w \in V$, $v \sim w$ denotes that v and w are neighbours in $\mathbb{T}$. We extend this notation for subsets of V : For $S, U \subset V$, $S \sim U$ denotes that $\exists s \in S, \exists u \in U$ such that $s \sim u$. Moreover, $S \nsim U$ denotes that $S \sim U$ does not hold.

Definition 5.4.1. *Let $n \in \mathbb{N}$. We say that a sequence of vertices $v^1, v^2, \ldots, v^n$, denoted by ρ, is a path if*

- $v^i \sim v^{i+1}$ *for* $i = 1, 2, \ldots, (n-1)$, *and*
- $v^i \neq v^j$ *when* $i \neq j$ *for* $i, j = 1, 2, \ldots, n$.

We say that ρ is non self touching, if $u, w \in \rho$ with $u \sim w$ then there is some $i \in \mathbb{N}$ with $1 \leq i \leq n-1$ such that either $u = v^i$ and $w = v^{i+1}$ or $u = v^{i+1}$ and $w = v^i$. We consider our paths to be ordered: v^1 is the starting point and v^n is the ending point of ρ. For $u, w \in \rho$ we say that u is after w in ρ, and denote it by $w \prec_\rho u$ if $u = v^i$ and $w = v^j$ for some $i, j \in \mathbb{N}$ with $1 \leq j < i \leq n$. For $u, w \in \rho$, $u \preceq_\rho w$ denotes that either $u = w$ or $u \prec_\rho w$. When it is clear from the context which path we are considering, we omit the subscript ρ. For $u, w, z \in \rho$ we say that w is in between u and z if $u \preceq w \preceq z$ or $u \succeq w \succeq z$. For $u, z \in \rho$ with $u \preceq_\rho z$ let $\rho_{u,z}$ denote the subpath of ρ consisting of the vertices between u and z.

We say that two paths ρ_1, ρ_2 are non-touching, if $\rho_1 \nsim \rho_2$.

Definition 5.4.2. *Let $n \in \mathbb{N}$ and sequence of vertices $v^1, v^2, \ldots, v^n$, satisfying*

- $v^i \sim v^{i+1 \mod n}$ *for* $i = 1, 2, \ldots, n$, *and*
- $v^i \neq v^j$ *when* $i \neq j$ *for* $i, j = 1, 2, \ldots, n$.

A loop ν is the equivalence class of the sequence $\left(v^1, v^2, \ldots, v^n\right)$ under cyclic permutations, i.e ν is the set of sequences $\left(v^j, v^{j+1 \mod n}, \ldots, v^{j+n-1 \mod n}\right)$ for $j = 1, 2, \ldots, n$. ν is non-self touching if for all $\left(w^1, w^2, \ldots, w^n\right) \in \nu$, the path $\left(w^1, w^2, \ldots, w^{n-1}\right)$ is non-self touching.

With a slight abuse of notation, we say that a loop ν contains a vertex v and denote it by $v \in \nu$ if $v = v^i$ for some $i \in \{1, 2, \ldots, n\}$. Let $v, w \in \nu$ with $v \neq w$ and let ρ denote the unique path which starts at v and represents ν. With the notation of Definition 5.4.1, let $\nu_{v,w} := \rho_{v,w}$ denote the arc of ν starting at v and ending at w.

5.4.2 Thick paths

Definition 5.4.3. *Let $M \in \mathbb{N}$ be fixed. The M-grid is the set of parallelograms $B\left((2M+1)z; M\right)$ for $z \in V$. Let π be a sequence consisting of some parallelograms of the M-grid. We say that π is an M-gridpath, if for any two consecutive parallelograms B, B' of π share a side, i.e $|\partial B \cap B'| \geq 2$.*

Definition 5.4.4. *Let C be a subgraph of $\mathbb{T}$, $D \subset V$ and $a, b \in \mathbb{N}$. We say that C is (a,b)-nice in D, if it satisfies the conditions*

1. *C is a connected induced subgraph of $\mathbb{T}$,*
2. *∂C is a disjoint union of non-touching loops, each with diameter bigger than $2b$.*
3. *Let $u, v \in \partial C \cap D$ with $d(u,v) \leq a$. Then u, v are contained in the same loop γ of ∂C, and $\operatorname{diam}(\gamma_{u,v}) \wedge \operatorname{diam}(\gamma_{v,u}) \leq b$.*

In the case where $D = V$, we say that C is (a,b)-nice.

Let C be (a,b)-nice for some $a, b \in \mathbb{N}$. Condition 3 of Definition 5.4.4, roughly speaking, says that if there is a corridor in C with width less than a, then it connects two parts of C such that one part has diameter at most b. This suggests that when b is small compared to $\operatorname{diam}(C)$, then we can move a parallelogram with side length $O(a)$ in C between two distant points of C. This intuitive argument leads us to the following lemma.

Lemma 5.4.5. *Let $a, b \in \mathbb{N}$ with $a \geq 2000$. Let C be an (a,b)-nice subgraph of $\mathbb{T}$. Then there is a $\lfloor a/200 - 10 \rfloor$-gridpath contained in C with diameter at least* $\operatorname{diam}(C) - 2b - 2a - 12$.

We use the following 'local' version of Lemma 5.4.5:

Lemma 5.4.6. *Let $a, b, c \in \mathbb{N}$ with $a \geq 2000$. Let C be subgraph of $\mathbb{T}$ which is (a,b)-nice in $B(c)$. Let C' be a connected component of $C \cap B(c)$. Then there is a $\lfloor a/200 - 10 \rfloor$-gridpath contained in C' with diameter at least* $\operatorname{diam}(C') - 2b - 2a - 12$.

Proof of Lemma 5.4.5 and 5.4.6. The proof of Lemma 5.4.5 and 5.4.6 have geometric/topologic nature, hence it is moved to Section 5.A.2 of the Appendix. □

We recall and prove Proposition 5.3.6 in the following.

Proposition 5.3.6. *Let* $\theta \in (0,1)$, $\varepsilon > 0$ *and* $\lambda_1 K, \in \mathbb{R}$. *Recall the notation* $FC(t, K+2, N)$ *from Lemma 5.3.3. There exists* $\lambda_2 = \lambda_2(\lambda_1, \theta, \varepsilon)$ *and* $N_0 = N_0(\lambda_1, \theta, \varepsilon)$ *such that the probability of the intersection of the events*

- $\exists v \in B(KN)$ *such that* $\operatorname{diam}(\mathcal{C}_a(v; \lambda_1, N)) \geq (1+\theta) N$, *and*
- *none of the clusters intersecting* $B((K+2)N)$ *freeze in the time interval* $(p_{\lambda_1}(N), p_{\lambda_2}(N)]$, *i.e.*

$$FC(p_{\lambda_1}(N), K+2, N) = FC(p_{\lambda_2}(N), K+2, N)$$

is less than ε for $N \geq N_0$.

Proof of Proposition 5.3.6. By Lemma 5.2.27 we choose $\lambda_0 = \lambda_0(\varepsilon, K) \leq \lambda_1$ and $N_1 = N_1(\varepsilon, K)$ such that

$$\mathbb{P}(\mathcal{N}_c(\lambda_0, 1/6, K+6, N)) > 1 - \varepsilon/3. \tag{5.4.1}$$

By Corollary 5.2.13 we choose $\eta < \theta/10$ and $N_2 = N_2(\eta, \theta, \lambda_0, \lambda_1, K)$ such that

$$\mathbb{P}(\mathcal{NA}(2\eta, \theta/10, \lambda_0, \lambda_1, K+4, N)) > 1 - \varepsilon/3 \tag{5.4.2}$$

for all $N \geq N_2$. Let

$$E := \mathcal{N}_c(\lambda_0, 1/6, K+6, N) \cap \mathcal{NA}(2\eta, \theta/10, \lambda_0, \lambda_1, K+4, N).$$

Claim 5.4.7. *Let* $u \in B(KN)$ *with* $\operatorname{diam}(\mathcal{C}_a(u; \lambda_1, N)) \geq (1+\theta) N$. *Then* $\mathcal{C}_a(u; \lambda_1, N)$ *is* $\left(\eta N, \frac{\theta}{10} N\right)$*-nice in* $B(u; 2N)$ *on the event* E.

Proof of Claim 5.4.7. Let us check the conditions of Definition 5.4.4. The Condition 1 is satisfied by the definition of $\mathcal{C}_a(u; \lambda_1, N)$.

All the holes of $\mathcal{C}_a(u; \lambda_1, N)$ contain a frozen cluster, which have diameter at least N. This combined with $2\frac{\theta}{10}N < N$, shows that Condition 2 of Definition 5.4.4 holds.

Let $x, y \in \partial\mathcal{C}_a(u; \lambda_1, K) \cap B(u; 2N)$ with $d(x,y) \leq \eta N$. We have two cases.

Case 1. x, y lie in different loops of $\partial\mathcal{C}_a(u; \lambda_1, N)$. For $i = x, y$, let γ_i denote the loop containing i. Furthermore, let $\tilde{\gamma}_i$ denote the connected component of i in $\gamma_i \cap B(i; 2N)$. We have $\operatorname{diam}(\tilde{\gamma}_i) \geq N$. Moreover, $\tilde{\gamma}_i \subset B(i; 2N) \subset B((K+4)N)$. Observation 5.3.2 gives that on the event $\mathcal{N}_c(\lambda_0, 1/6, K+6, N)$, $\tilde{\gamma}_i$ is $p_{\lambda_0}(N)$-closed. Hence each of $\tilde{\gamma}_x$ and $\tilde{\gamma}_y$ gives two closed $p_{\lambda_0}(N)$-closed arms in $A(x; 2\eta N, N/2)$. Moreover, the frozen clusters neighbouring x and y provide two disjoint $p_{\lambda_1}(N)$-open arms. Hence there are 6 disjoint arms in $A(x; 2\eta N, N/2)$, thus

$$\mathcal{NA}^c(2\eta, \theta/10, \lambda_0, \lambda_1, K+4, N)$$

occurs.

Case 2. x, y lie on the same loop of $\partial\mathcal{C}_a(u;\lambda_1,N)$. This case can be treated similarly to Case 1, with the difference that if x, y violate Condition 3 of Definition 5.4.4 then we get 6 arms in $A\left(x; 2\eta N, \frac{\theta}{10}N\right)$. Hence

$$\mathcal{N}\mathcal{A}^c\left(2\eta, \theta/10, \lambda_0, \lambda_1, K+4, N\right)$$

occurs.

Hence in both cases E^c occurs. Thus on the event E all the conditions of Definition 5.4.4 are satisfied for $\mathcal{C}_a(u;\lambda_1,N)$, which finishes the proof of Claim 5.4.7. □

Let us turn back to the proof of Proposition 5.3.6. Let $u \in B(KN)$ with $\operatorname{diam}(\mathcal{C}_a(u;\lambda_1,N)) \geq (1+\theta)N$. Let $\tilde{\mathcal{C}}_a(u,\lambda_1,N)$ denote the connected component of u in $\mathcal{C}_a(u,\lambda_1,N) \cap B(u;2N)$. Since $\operatorname{diam}(\mathcal{C}_a(u;\lambda_1,N)) \geq (1+\theta)N$ and $\theta < 1$, we have $\operatorname{diam}\left(\tilde{\mathcal{C}}_a(u;\lambda_1,N)\right) \geq (1+\theta)N$. By Lemma 5.4.6 we set $\eta = \eta(\theta) \in (0, \theta/100)$ and $N_3 = N_3(\theta)$ such that on the event E for all $u \in B(KN)$, with $\operatorname{diam}(\mathcal{C}_a(u;\lambda_1,N)) \geq (1+\theta)N$ there is a $\lfloor \eta N \rfloor$-gridpath $\rho_u \subset \tilde{\mathcal{C}}_a(u;\lambda_1,N)$ with $\operatorname{diam}(\rho_u) \geq (1+\theta/2)N$ for $N \geq N_3$.

Lemma 5.2.27 gives that there is $\lambda_2 = \lambda_2(\varepsilon,\eta,K)$ and $N_4 = N_4(\varepsilon,\eta,K)$ such that

$$\mathbb{P}\left(\mathcal{N}_o\left(\lambda_2, \eta/2, K+4, N\right)\right) > 1 - \varepsilon/3 \tag{5.4.3}$$

for $N \geq N_4(\varepsilon,\eta,K)$. We set $N_0 := \bigvee_{i=1}^4 N_i$. Let

$$\begin{aligned} G :=& E \cap \mathcal{N}_o\left(\lambda_2, \eta/2, K+4, N\right), \\ M :=& \left\{\exists v \in B(KN) \text{ s.t. } \operatorname{diam}\left(\mathcal{C}_a(v;\lambda_1,N)\right) \geq (1+\theta)N\right\} \cap G. \end{aligned}$$

Combination of (5.4.1), (5.4.2) and (5.4.3) gives that

$$\mathbb{P}(G^c) < \varepsilon \tag{5.4.4}$$

for $N \geq N_0$.

Recall that for $N \geq N_0$, on the event E for $u \in B(KN)$, with $\operatorname{diam}(\mathcal{C}_a(u;\lambda_1,N)) \geq (1+\theta)N$ there is a $\lfloor \eta N \rfloor$-gridpath $\rho_u \subset \tilde{\mathcal{C}}_a(u;\lambda_1,N)$ with diameter at least $(1+\theta/2)N$. On the event $\mathcal{N}_o(\lambda_2,\eta/2,K+4,N)$, this gridpath ρ_u is a subset of $B((K+2)N)$ contains a $p_{\lambda_2}(N)$-open component with diameter at least N. Hence on the event M, at least one cluster intersecting $B((K+2)N)$ freezes in the time interval $(p_{\lambda_1}(N), p_{\lambda_2}(N)]$. That is

$$M \subseteq \left\{FC\left(p_{\lambda_1}(N), K+2, N\right) < FC\left(p_{\lambda_2}(N), K+2, N\right)\right\}.$$

Thus

$$\begin{aligned} &\left\{\exists v \in B(KN) \text{ s.t. } \operatorname{diam}\left(\mathcal{C}_a(v;\lambda_1,N)\right) \geq (1+\theta)N\right\} \\ &\qquad \cap \left\{FC\left(p_{\lambda_1}(N), K+2, N\right) = FC\left(p_{\lambda_2}(N), K+2, N\right)\right\} \subset G^c, \end{aligned}$$

which together with (5.4.4) finishes the proof of Proposition 5.3.6. □

5.5 Proof of Proposition 5.3.5

5.5.1 Lowest point of the lowest crossing in parallelograms

Recall the notation of Section 5.4.1.

Definition 5.5.1. *Let R be a connected subgraph of $\mathbb{T}$ and let $r \subset \partial R$. We define $\mathcal{L}(R,r)$ as the (random) set of lowest vertices $v \in R$ such that v is closed, and there are two non-touching closed paths in R starting at a vertex neighbouring to v and ending at r.*

Consider the site percolation model on the triangular lattice with parameter $p \in [0,1]$. We investigate the distribution of $\mathcal{L}(R,r)$ in the case where $p = p_\lambda(N)$, $R = B(bN)$ and $r = top(B(bN)) := [-bN, bN] \boxtimes \{\lfloor bN \rfloor + 1\}$ for $\lambda \in \mathbb{R}$ and $b > 0$.

Definition 5.5.2. *For a parallelogram B, let $HCr(B)$ denote set of paths in B which connect the left and the right sides of B. For $\rho \in HCr(B)$, let $Be(\rho) = Be(\rho, B)$ denote the set of vertices in B which are 'under' ρ. It is the set of vertices $v \in B \setminus \rho$ which are connected to the bottom side of B. Furthermore, we define $Ab(\rho) = Ab(\rho, B) := B \setminus (\rho \cup Be(\rho, B))$.*

Lemma 5.5.3. *Let $a, b \in (0,1)$ with $5a < b$. For $k, l, N \in \mathbb{N}$ with $l < k$ we define the parallelogram*

$$B_{l,k} := [-aN, aN] \boxtimes \left(\left(2\frac{l}{k} - 1\right) aN, \left(2\frac{l+1}{k} - 1\right) aN \right]$$

and the event

$$L_{l,k} =: \{\mathcal{L}(B(bN), top(bN)) \cap B_{l,k} \neq \emptyset\}.$$

In words $L_{l,k}$ is the event that at least one of the lowest vertices of $B(bN)$ with two non-touching closed paths $B(bN)$ to the top side of $B(bN)$ is in the parallelogram $B_{l,k}$.

Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Then there exist $C = C(a, b, \lambda_1, \lambda_2)$ and $N_0 = N_0(a, b, \lambda_1, \lambda_2, k)$ such that for all $\lambda \in [\lambda_1, \lambda_2]$ and $k, l \in \mathbb{N}$ with $l \leq k-1$ we have

$$\mathbb{P}_{p_\lambda(N)}(L_{l,k}) \leq Ck^{-1} \tag{5.5.1}$$

for $N \geq N_0$. In particular, the upper bound in (5.5.1) is uniform in l.

Proof of Lemma 5.5.3. Note that $L_{l,k} \cap L_{m,k} = \emptyset$ when $l \neq m$. Hence it is enough to show that there exist $c = c(a, b, \lambda_1, \lambda_2) > 0$ and $N_0 = N_0(a, b, \lambda_1, \lambda_2)$ such that for all $l, m \in [0, k-1] \cap \mathbb{Z}$ with $0 \leq l \leq k-1$ and $m \leq k-2$ we have

$$c\mathbb{P}_{p_\lambda(N)}(L_{l,k}) \leq \mathbb{P}_{p_\lambda(N)}(L_{m,k} \cup L_{m+1,k}) \tag{5.5.2}$$

for $N \geq N_0$.

Let k, l be given. Let s_L (s_R) denote the left (right) endpoint of $top(bN)$. We say that a path $\rho \subseteq B(bN) \cup top(bN)$ is good, if it

- starts at s_L and ends at s_R,
- it is non-self touching
- and one of its lowest points is in $B_{l,k}$.

Let γ denote the lowest non-self touching path in $B(bN) \cup top(bN)$ which starts at s_L and ends at s_R, and of which all the vertices outside of $top(bN)$ are closed. On the event $L_{l,k}$ γ is good.

The following event plays a crucial role in the proof. Let ρ be a fixed good path. Recall Definition 5.5.2 and let $Be(\rho) := Be(\rho, B(bN))$. Let O_ρ denote the event that there is path ν such that

- $\nu \subseteq B_0 := [-bN, bN] \boxtimes \left[-bN, \frac{b}{4}N\right]$,
- ν connects the left and the right sides of the parallelogram
$$B_1 := [-bN, bN] \boxtimes \left[aN, \frac{b}{4}N\right],$$
- ν is a concatenation of some open paths which lie in $Be(\rho) \cap B_1$, and of some subpaths of ρ.

Clearly, O_ρ is an increasing event. On O_ρ, let $\xi(\rho)$ denote the lowest path which satisfies the conditions in the definition of O_ρ.

Case 1. First we consider the case where $l < m$. Recall the definition of increasing events from Definition 5.2.1. For any good path ρ, on the event that all of the vertices of $\rho \setminus top(bN)$ are closed, the event $\{\gamma = \rho\}$ is increasing in the configuration in $B(bN) \setminus \rho$. Let $H_1 := \mathcal{H}_o([-bN, bN] \boxtimes [-bN, -(b-a)N])$. The events H_1 and O_ρ are increasing. Thus by FKG and by Lemma 5.2.20, we have

$$\begin{aligned}
&\mathbb{P}_{p_\lambda(N)}(L_{l,k} \cap H_1 \cap O_\gamma) \\
&\quad = \sum_{\rho \text{ good}} \mathbb{P}_{p_\lambda(N)}(H_1 \cap O_\rho \cap \{\gamma = \rho\} \mid \rho \setminus top(bN) \text{ is closed}) \\
&\qquad\qquad\qquad\qquad \mathbb{P}_{p_\lambda(N)}(\rho \setminus top(bN) \text{ is closed}) \\
&\quad \geq \sum_{\rho \text{ good}} \mathbb{P}_{p_\lambda(N)}(\{\gamma = \rho\} \mid \rho \setminus top(bN) \text{ is closed}) \\
&\qquad \mathbb{P}_{p_\lambda(N)}(H_1 \mid \rho \setminus top(bN) \text{ is closed}) \mathbb{P}_{p_\lambda(N)}(O_\rho \mid \rho \setminus top(bN) \text{ is closed}) \\
&\qquad\qquad\qquad\qquad \mathbb{P}_{p_\lambda(N)}(\rho \setminus top(bN) \text{ is closed}) \\
&\quad \geq \sum_{\rho \text{ good}} c_1 \mathbb{P}_{p_\lambda(N)}(\{\gamma = \rho\} \mid \rho \setminus top(bN) \text{ is closed}) \\
&\qquad\qquad\qquad\qquad \mathbb{P}_{p_\lambda(N)}(\rho \setminus top(bN) \text{ is closed}) \\
&\quad = c_1 \mathbb{P}_{p_\lambda(N)}(L_{l,k})
\end{aligned} \tag{5.5.3}$$

for $c_1 = c_1(a, b, \lambda_1, \lambda_2) > 0$ and for $N \geq N_1 = N_1(a, b, \lambda_1, \lambda_2)$.

There is $N_2 = N_2(k)$ such that for $N \geq N_2$ and for all $l, m \in [0, k-1] \cap \mathbb{Z}$ with $l < m$ there is a shift $S = S(l, m, k)$ which moves the parallelogram $B_{l,k}$

to a subset of $B_{m-1,k} \cup B_{m,k}$. Let us take a configuration in $\omega \in \{o,c\}^V$ for which the event $L_{l,k} \cap H_1 \cap O_\gamma$ holds. Then shifted configuration $S(\omega)$ satisfies $L_{m-1,k} \cup L_{m,k}$. Hence

$$\begin{aligned}\mathbb{P}_{p_\lambda(N)}\left(L_{m-1,k} \cup L_{m,k}\right) &\geq \mathbb{P}_{p_\lambda(N)}\left(L_{l,k} \cap H_1 \cap O_\gamma\right), \\ &\geq c_1 \mathbb{P}_{p_\lambda(N)}\left(L_{l,k}\right) \end{aligned} \tag{5.5.4}$$

by (5.5.3) for $N \geq N_1 \vee N_2$.

Case 2. When $l > m$ we have a similar argument. The difference is that now we want to shift downwards. To get a configuration in $L_{m,k}$ after the shift, we have to extend the closed path γ see Figure 5.1 for more details.

Let ρ be a good path. Recall the definition of decreasing events from Definition 5.2.1, the definition of γ, O_ρ and $\zeta(\rho)$ from above. Let us condition on the event that all the vertices of $\rho \setminus top(bN)$ are closed. Then the event $\{\gamma = \rho\}$ is increasing on the configuration in $B(bN) \setminus \rho$, and it only depends on the configuration in $Be(\rho)$. Hence a combination of FKG and Corollary 5.2.20 give that

$$\begin{aligned}
&\mathbb{P}_{p_\lambda(N)}\left(L_{l,k} \cap O_\gamma\right) \\
&\quad= \sum_{\rho \text{ good}} \mathbb{P}_{p_\lambda(N)}\left(O_\rho \cap \{\gamma = \rho\} \mid \rho \setminus top(bN) \text{ is closed}\right) \\
&\qquad\qquad \mathbb{P}_{p_\lambda(N)}\left(\rho \setminus top(bN) \text{ is closed}\right) \\
&\quad\geq \sum_{\rho \text{ good}} \mathbb{P}_{p_\lambda(N)}\left(\{\gamma = \rho\} \mid \rho \setminus top(bN) \text{ is closed}\right) \\
&\qquad\qquad \mathbb{P}_{p_\lambda(N)}\left(O_\rho \mid \rho \setminus top(bN) \text{ is closed}\right) \mathbb{P}_{p_\lambda(N)}\left(\rho \setminus top(bN) \text{ is closed}\right) \\
&\quad\geq \sum_{\rho \text{ good}} \mathbb{P}_{p_\lambda(N)}\left(\{\gamma = \rho\} \mid \rho \setminus top(bN) \text{ is closed}\right) \\
&\qquad\qquad \mathbb{P}_{p_\lambda(N)}\left(\mathcal{H}_o(B_1)\right) \mathbb{P}_{p_\lambda(N)}\left(\rho \setminus top(bN) \text{ is closed}\right) \\
&\quad\geq c_2(\lambda_1, \lambda_2, a, b)\, \mathbb{P}_{p_\lambda(N)}\left(L_{l,k}\right)
\end{aligned} \tag{5.5.5}$$

for $c_2 = c_2(a, b, \lambda_1, \lambda_2) > 0$ and for $N \geq N_3 = N_3(a, b, \lambda_1, \lambda_2)$.

For $W \subseteq V$ and $\omega \in \{o,c\}^V$, $\omega_W \in \{o,c\}^W$ denotes the restriction of ω to the configuration in W. That is $\omega_W(v) = \omega(v)$ for $v \in W$. Recall Definition 5.5.2. Let $\zeta \in HCr(B_0)$ be arbitrary. It is easy to check that the event $L_{l,k} \cap O_\gamma \cap \{\xi(\gamma) = \zeta\}$ is decreasing in the configuration in $Ab(\zeta)$. Let us take the parallelograms $B_2 = [-bN, bN] \boxtimes \left[\frac{b}{4}N, \frac{b}{2}N\right]$, $B_3 = [-bN, bN] \boxtimes \left[\frac{3}{4}bN, bN\right]$, $B_4 = \left[-bN, -\frac{1}{2}bN\right] \boxtimes \left[\frac{1}{4}bN, (b+2a)N\right]$ and $B_5 = \left[\frac{1}{2}bN, bN\right] \boxtimes \left[\frac{1}{4}bN, (b+2a)N\right]$. Let $\mathcal{D} = \mathcal{H}_c(B_2) \cap \mathcal{H}_c(B_3) \cap \mathcal{V}_c(B_4) \cap \mathcal{V}_c(B_5)$. Clearly, $\mathcal{D}$ is a decreasing event. Hence a combination of FKG and Corollary 5.2.20 give that for $c_3 = c_3(a, b, \lambda_1, \lambda_2) > 0$ and $N \geq N_4 = N_4(a, b, \lambda_1, \lambda_2)$ we have

$$\begin{aligned}
&\mathbb{P}_{p_\lambda(N)}\left(L_{k,l} \cap O_\gamma \cap \mathcal{D}\right) \\
&\quad= \sum_\zeta \sum_\sigma \mathbb{P}_{p_\lambda(N)}\left(L_{k,l} \cap O_\gamma \cap \{\xi(\gamma) = \zeta\} \cap \mathcal{D} \mid \omega_{\zeta \cup Be(\zeta)} = \sigma\right) \\
&\qquad\qquad \mathbb{P}_{p_\lambda(N)}\left(\omega_{\zeta \cup Be(\zeta)} = \sigma\right) \\
&\quad\geq \sum_\zeta \sum_\sigma \mathbb{P}_{p_\lambda(N)}\left(L_{k,l} \cap O_\gamma \cap \{\xi(\gamma) = \zeta\} \mid \omega_{\zeta \cup Be(\zeta)} = \sigma\right) \\
&\qquad\qquad \mathbb{P}_{p_\lambda(N)}\left(\mathcal{D} \mid \omega_{\zeta \cup Be(\zeta)} = \sigma\right) \mathbb{P}_{p_\lambda(N)}\left(\omega_{\zeta \cup Be(\zeta)} = \sigma\right)
\end{aligned}$$

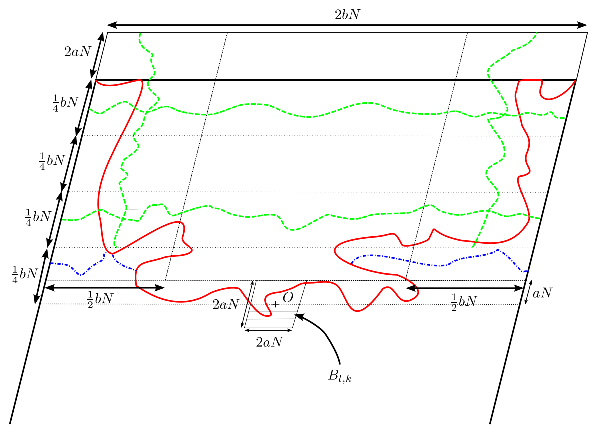

Figure 5.1: The continuous line represents γ. The dashed paths are the closed crossings of $\mathcal{D}$, which allow us to prolong γ. The dashed-dotted paths are the open parts of $\xi(\gamma)$. They, together with γ, prevent the occurrence of closed vertices below the lowest point of γ with two closed arms to the top side after the shift.

$$\begin{aligned}
&= \sum_{\zeta}\sum_{\sigma} \mathbb{P}_{p_\lambda(N)}\left(L_{k,l}\cap O_\gamma \cap \{\xi(\gamma)=\zeta\} \mid \omega_{\zeta\cup Be(\zeta)}=\sigma\right) \\
&\qquad\qquad\qquad \mathbb{P}_{p_\lambda(N)}(\mathcal{D})\,\mathbb{P}_{p_\lambda(N)}\left(\omega_{\zeta\cup Be(\zeta)}=\sigma\right) \\
&\geq c_3(a,b,\lambda_1,\lambda_2)\sum_{\zeta}\sum_{\sigma} \mathbb{P}_{p_\lambda(N)}\left(L_{k,l}\cap O_\gamma \cap \{\xi(\gamma)=\zeta\} \mid \omega_{\zeta\cup Be(\zeta)}=\sigma\right) \\
&\qquad\qquad\qquad \mathbb{P}_{p_\lambda(N)}\left(\omega_{\zeta\cup Be(\zeta)}=\sigma\right) \\
&= c_3(a,b,\lambda_1,\lambda_2)\,\mathbb{P}_{p_\lambda(N)}\left(L_{k,l}\cap O_\gamma\right) \qquad (5.5.6)
\end{aligned}$$

where the summation in ζ is over $HCr(B_0)$ and the summation in σ is over $\{o,c\}^{\zeta\cup Be(\zeta)}$. In the third line we used that $\mathcal{D}$ does not depend on the configuration in $\zeta\cup Be(\zeta)$.

There is $N_5 = N_5(k)$ such that for $N \geq N_5$ and for all $l,m \in [0,k-1]\cap\mathbb{Z}$ with $l > m$ there is a shift $S = S(l,m,k)$ which moves the parallelogram $B_{l,k}$ to a subset of $B_{m,k}\cup B_{m+1,k}$. Let us take a configuration $\omega \in \{o,c\}^V$ which satisfies $L_{k,l}\cap O_\gamma\cap\mathcal{D}$. Then the shifted configuration $S(\omega)$ satisfies $L_{m,k}\cup L_{m+1,k}$, see Figure 5.1 for more details. Hence for $N \geq N_3 \vee N_4 \vee N_5$ we have

$$\begin{aligned}
\mathbb{P}_{p_\lambda(N)}\left(L_{m,k}\cup L_{m+1,k}\right) &\geq \mathbb{P}_{p_\lambda(N)}\left(L_{k,l}\cap O_\gamma\cap\mathcal{D}\right) \\
&\geq c_2c_3\mathbb{P}_{p_\lambda(N)}\left(L_{l,k}\right) \qquad (5.5.7)
\end{aligned}$$

by a combination of (5.5.5) and (5.5.6).

(5.5.4) together with (5.5.7) implies (5.5.2) and finishes the proof of Lemma 5.5.3. □

Remark 5.5.4. Let $a,b,\lambda,\lambda_1,\lambda_2$ be as in Lemma 5.5.3. Standard RSW techniques give that there is $c' = c'(a, b, \lambda_1, \lambda_2) > 0$ and $N_0 = N_0(a, b, \lambda_1, \lambda_2)$ such that

$$\mathbb{P}_{p_\lambda(N)}\left(\mathcal{L}\left(B\left(bN\right), t\left(bN\right)\right) \cap B\left(aN\right) \neq \emptyset\right) \geq c'$$

for $N \geq N_0$. This combined with arguments similar to the proof of Lemma 5.5.3 we get that there is $C' = C'(a, b, \lambda_1, \lambda_2) > 0$ and $N_1 = N_1(a, b, \lambda_1, \lambda_2, k)$ such that

$$\mathbb{P}_{p_\lambda(N)}\left(L_{l,k}\right) \geq C' k^{-1}$$

for $N \geq N_1$. This, together with Lemma 5.5.3, implies that that $\mathbb{P}_{p_\lambda(N)}\left(L_{l,k}\right) = O\left(k^{-1}\right)$.

5.5.2 Lowest point of the lowest crossing in regular regions

Recall Definition 5.5.1. Let $B \subset B'$ be parallelograms, and let R be a subgraph of $\mathbb{T}$ with $B \subset R \subset B'$. Furthermore let $r \subset \partial R$. Our next aim is to compare the event $\mathcal{L}(R, r) \cap B \neq \emptyset$ to $\mathcal{L}(B', top(B')) \cap B \neq \emptyset$ in the case where the pair (R, r) is 'regular'. We make this precise in the following.

We say that a subgraph $H \subseteq \mathbb{T}$ is simply connected, if it is connected and for all loops $\sigma \subseteq H$, all of the finite components of $\mathbb{T} \setminus \sigma$ are contained in H.

Definition 5.5.5. *Let $a, b \in \mathbb{N}$ such that $5a < b$. A pair (R, r) is (a, b)-regular, if*

1. *R is an induced subgraph of $\mathbb{T}$ such that $cl(R)$ is simply connected,*
2. *$B(a) \subseteq R \subseteq B(b)$,*
3. *$r \subset \partial R$, such that $\emptyset \neq r \subsetneq \partial R$. Furthermore, r and $\partial R \backslash r$ are self-avoiding paths such that R is on the right hand side, as we walk along them.*
4. *$r \subseteq [-b, b] \boxtimes [5a, b]$.*

Lemma 5.5.6. *Let $a, b \in (0, 1)$ with $5a < b$ and $\lambda \in \mathbb{R}$. Let (R, r) be (aN, bN)-regular. For $k, l, N \in \mathbb{N}$ with $l < k$ we define the events*

$$L_{l,k}\left(B\left(2bN\right), top\left(B\left(2bN\right)\right)\right) := \left\{\mathcal{L}\left(B\left(2bN\right), top\left(2bN\right)\right) \cap B_{l,k} \neq \emptyset\right\},$$
$$L_{l,k}\left(R, r\right) := \left\{\mathcal{L}\left(R, r\right) \cap B_{l,k} \neq \emptyset\right\},$$

where

$$B_{l,k} := [-aN, aN] \boxtimes \left(\left(2\frac{l}{k} - 1\right) aN, \left(2\frac{l+1}{k} - 1\right) aN\right].$$

Let $\lambda_1, \lambda_2 \in \mathbb{R}$. Then there exist $C = C(a, b, \lambda_1, \lambda_2)$ and $N_0 = N_0(a, b, \lambda_1, \lambda_2, k)$ such that for all $\lambda \in [\lambda_1, \lambda_2]$ and $k, l \in \mathbb{N}$ with $l \leq k - 1$ we have

$$\mathbb{P}_{p_\lambda(N)}\left(L_{l,k}\left(R, r\right)\right) \leq C \mathbb{P}_{p_\lambda(N)}\left(L_{l,k}\left(B\left(2bN\right), top\left(B\left(2bN\right)\right)\right)\right) \tag{5.5.8}$$

for $N \geq N_0$.

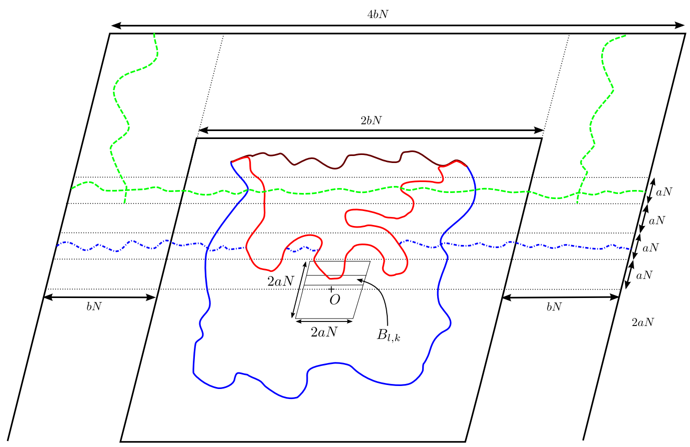

Figure 5.2: The dashed paths are the closed crossings of the event $\mathcal{D}$ which allow us to prolong γ. The dashed-dotted paths are the open parts of $\xi(\gamma)$. They, together with γ, prevent the occurrence of closed vertices below the lowest point of γ with two closed arms to the top side of $B(2bN)$.

Proof of Lemma 5.5.6. The proof follows the arguments of Case 2 in the proof of Lemma 5.5.3. Our aim is to show that, conditioning on $L_{l,k}(R,r)$, the open and closed crossings of Figure 5.2 occur with probability bounded away from 0 cf. Figure 5.1.

Let k, l be given. Let s_L (s_R) denote the starting (ending) vertex of r. We say that a path $\rho \subseteq R \cup r$ is good, if it

- starts at s_L and ends at s_R,
- it is non-self touching
- and one of its lowest points is in $B_{l,k}$.

Let ρ be a fixed good path. Let $Be(\rho, R)$ denote the set of vertices in R 'under' ρ. It is the intersection of R with the connected component of $\partial R \setminus r$ in $cl(R) \setminus \rho$. Let $Ab(\rho, R) := R \setminus Be(\rho, R)$. Recall Definition 5.5.2.

Let O_ρ denote the event that there is path ν such that

- ν is non self-touching,
- $\nu \subseteq B_0 := [-2bN, 2bN] \boxtimes [-aN, 2aN]$,
- ν connects the left and the right side of the parallelogram
$$B_1 := [-2bN, 2bN] \boxtimes [aN, 2aN],$$
- $\nu \setminus R \subset B_1$ and the vertices in $\nu \setminus R$ are open,

- each of the paths of $\nu \cap R$ is a concatenation of some open paths which lie in $Be(\rho, B(bN)) \cap B_1$, and of some subpaths of ρ.

Let γ denote the lowest non-self touching path in $R \cup r$ which starts at s_L and ends at s_R, and of which all the vertices outside of r are closed. Note that on the event $L_{l,k}(R,r)$, γ is good. By simple modifications of the arguments of Case 2 in the proof of Lemma 5.5.3 we get that there are $c_1 = c_1(a,b,\lambda_1,\lambda_2) > 0$ and $N_1 = N_1(a,b,\lambda_1,\lambda_2)$ such that

$$\mathbb{P}_{p_\lambda(N)}(L_{l,k}(R,r) \cap O_\gamma) \geq c_1 \mathbb{P}_{p_\lambda(N)}(L_{l,k}(R,r)) \tag{5.5.9}$$

for $l,k \in \mathbb{N}$, $0 \leq l \leq k-1$, $\lambda \in [\lambda_1,\lambda_2]$ for $N \geq N_1$.

Recall Definition 5.5.2. Let $\zeta \in HCr(B(2bN))$. On the event $L_{l,k}(R,r) \cap O_\gamma$ we have $R \cap (\mathbb{Z} \boxtimes [3aN, bN]) \subset Ab(\xi(\gamma), B(2bN))$. Hence the event $L_{l,k}(R,r) \cap O_\gamma \cap \{\xi(\gamma) = \zeta\}$ is decreasing on the configuration in $Ab(\zeta, B(2bN))$. Let $B_2 = [-2bN, 2bN] \boxtimes [3aN, 4aN]$, $B_3 = [-2bN, -bN] \boxtimes [3aN, 2bN]$, $B_4 = [bN, 2bN] \boxtimes [3aN, 2bN]$ and $\mathcal{D} = \mathcal{H}_c(B_2) \cap \mathcal{V}_c(B_3) \cap \mathcal{V}_c(B_4)$. The arguments of Case 2 of Lemma 5.5.3 give that there exist $c_2 = c_2(a,b,\lambda_1,\lambda_2) > 0$ and $N_2 = N_2(a,b,\lambda_1,\lambda_2,k)$ such that

$$\mathbb{P}_{p_\lambda(N)}(L_{l,k}(R,r) \cap O_\gamma \cap \mathcal{D}) \geq c_2 \mathbb{P}_{p_\lambda(N)}(L_{l,k}(R,r) \cap O_\gamma) \tag{5.5.10}$$

for $l,k \in \mathbb{N}$, $0 \leq l \leq k-1$, $\lambda \in [\lambda_1,\lambda_2]$ for $N \geq N_2$. Note that $L_{l,k}(R,r) \cap O_\gamma \cap \mathcal{D} \subset L_{l,k}(B(2bN), top(2bN))$. See Figure 5.2 for more details. This combined with (5.5.9) and (5.5.10) finishes the proof of Lemma 5.5.6. □

A combination of Lemma 5.5.3 and 5.5.6 gives the following:

Corollary 5.5.7. *Suppose that the conditions of Lemma 5.5.3 hold. Then there exist $c = c(a,b,\lambda_1,\lambda_2)$ and $N_0 = N_0(a,b,\lambda_1,\lambda_2,k)$ such that*

$$\mathbb{P}_{p_\lambda(N)}(L_{l,k}(R,r)) \leq ck^{-1}$$

for $l = 0,1,\ldots,k-1$, $\lambda \in [\lambda_1,\lambda_2]$ and $N \geq N_0$.

5.5.3 The diameter of the active clusters close to time 1/2

We turn to the N-parameter frozen percolation process. In the introduction we indicated that the N-parameter frozen percolation process exists since it is a finite range interacting particle system. It is also true that the process is measurable with respect to the τ values. The following lemma follows from the arguments in the second lecture of [38].

Lemma 5.5.8. *For $N \in \mathbb{N}$, the N-parameter frozen percolation process is adapted to the filtration generated by the random variables $\tau_v, v \in V$.*

Recall the notation $\mathcal{C}_a(v;\lambda)$ from Definition 5.3.4. We prove the following proposition.

Proposition 5.3.5. *For all $\lambda \in \mathbb{R}$ and $\varepsilon, K, \alpha > 0$, there exist $\theta = \theta(\lambda, \alpha, \varepsilon, K) > 0$ and $N_0 = N_0(\lambda, \alpha, \varepsilon, K)$ such that*

$$\mathbb{P}_N \left(\exists v \in B(KN) \ \ s.t. \ \ \mathrm{diam}\left(\mathcal{C}_a(v;\lambda)\right) \in \left((\alpha-\theta) N, (\alpha+\theta) N\right)\right) < \varepsilon \quad (5.5.11)$$

for $N \geq N_0$.

Proof of Proposition 5.3.5. Due to the length of the proof, we first give an outline. Let $\lambda, \varepsilon, K, \alpha$ as in the statement of Proposition 5.3.5.

For simplicity, we only give a sketch which shows that we can choose $\theta \in \left(0, \frac{1\wedge\alpha}{2}\right)$ such that

$$\mathbb{P}_N \left(\mathrm{diam}\left(\mathcal{C}_a(\lambda)\right) \in \left((\alpha-\theta) N, (\alpha+\theta) N\right)\right) < \varepsilon \quad (5.5.12)$$

for large N.

Let us denote by $\tilde{x}, \tilde{y}$ a pair of sites in the active cluster of the origin for which $d(\tilde{x}, \tilde{y}) = \mathrm{diam}\left(\mathcal{C}_a(\lambda)\right)$. We consider the case where $\tilde{x}$ is one of the lowest and $\tilde{y}$ is one of the highest vertices of the active cluster. The other case where the diameter is achieved as a distance between a leftmost and rightmost vertex can be treated in a similar way. Let x (y) denote a vertex which is a neighbour of $\tilde{x}$ $(\tilde{y})$, and lies below (above) it. Note that x and y are closed frozen vertices at time $p_\lambda(N)$.

In Step 1 we apply Observation 5.3.2 and Lemma 5.2.27 to set λ_0 so that with probability close to 1, there are no frozen clusters at time $p_{\lambda_0}(N)$ in $B((\alpha+2)N)$. Hence in the case where $\lambda_0 \geq \lambda$ the statement of Proposition 5.3.5 is trivial. Thus we can assume that $\lambda_0 < \lambda$, and the event in (5.5.11) is non-empty. We investigate the configuration close to x. In Step 2, we show that with probability close to 1, there is a unique frozen cluster F close to x. By Step 1, we can assume that it froze at time $p_{\lambda_F}(N)$ for $\lambda_F \in [\lambda_0, \lambda]$. In Step 4, we show that with probability close to 1, there is a graph $\mathcal{R} \subseteq \mathbb{T}$ such that its boundary consists of a $p_{\lambda_F}(N)$-closed arc, denoted by r_c, and a $p_{\lambda_F}(N)$-open arc. In Step 3,5 and 6 we show that with probability close to 1, we can impose some extra conditions on $\mathcal{R}$ and r_c and on the configuration in $\mathcal{R}$. We get a pair $(\mathcal{R}, r_c)$ with the following properties:

- $\partial\mathcal{R}$ is a certain outermost circuit, which is measurable with respect to the τ-values in $\mathbb{T} \setminus \mathcal{R}$, (Step 4)
- x is one of the lowest vertices of $\mathcal{R}$ with two non-touching $p_{\lambda_F}(N)$-closed arms in $\mathcal{R}$ to r_c, (Step 4)
- no matter how we change the τ values in $\mathcal{R}$, the N-parameter frozen percolation outside $\mathcal{R}$ does not change up to time $p_\lambda(N)$, (Step 3)
- satisfies a technical condition (Step 5)
- $y \in \mathbb{T} \backslash cl(\mathcal{R})$ (Step 4).

Let us condition on the τ-values in $\mathbb{T} \setminus \mathcal{R}$. The first and the third property of $(\mathcal{R}, r_c)$ implies that at time $p_{\lambda_F}(N)$, the vertices in $\mathcal{R}$ are open with probability $p_{\lambda_F}(N)$ and closed with probability $1 - p_{\lambda_F}(N)$ *independently* from each other. This combined with $y \in \mathbb{T} \setminus \mathcal{R}$ allows us to decouple the locations of x and y. Since $d(\tilde{x}, \tilde{y}) = \operatorname{diam}(\mathcal{C}_a(\lambda))$, to prove (5.5.12), it is enough to show that the second coordinate of x is not concentrated when we condition on the configuration in $\mathbb{T} \setminus \mathcal{R}$. We would like to use Corollary 5.5.7 for the pair $(\mathcal{R}, r_c)$. Unfortunately, this pair $(\mathcal{R}, r_c)$ might not satisfy all the conditions of Definition 5.5.5. To solve this problem we use the technical condition of Step 5 and we construct the pair $\left(\tilde{\mathcal{R}}, \tilde{r}_c\right)$ from $(\mathcal{R}, r_c)$ using a deterministic procedure in Step 6 such that

- $\tilde{\mathcal{R}} \subset \mathcal{R}$,
- a translated version of $\left(\tilde{\mathcal{R}}, \tilde{r}_c\right)$ is $(\alpha_3 N, \alpha_2 N)$-regular as of Definition 5.5.5 for some $\alpha_2, \alpha_3 > 0$, and
- x is one of the lowest vertices of $\tilde{\mathcal{R}}$ with two non-touching $p_{\lambda_F}(N)$-closed arms in $\tilde{\mathcal{R}}$ to $\tilde{r}_c$.

We apply Corollary 5.5.7 to $\left(\tilde{\mathcal{R}}, \tilde{r}_c\right)$ and get the required non-concentration result and finish the proof of Proposition 5.3.5. We make this argument precise in Step 7.

Remark. The structure of the proofs in Step 2-6 is an arm event hunting procedure. We take a some small neighbourhood of x. We deduce that if the required condition is violated, then certain mixed near-critical arm events or crossing events of thin parallelograms occur. These events have upper bounds with exponents strictly larger than 2. This implies that by choosing the neighbourhood small enough, we can set the probabilities of the events above as small as we want. In particular, we set the size of the neighborhood so small that the probability of the event where the condition of the step is not satisfied is as small as required, and finishes the proof of the step.

Let us turn to the precise proof.

Step 1. *We set λ_0 such that with probability close to* 1, *at time $p_{\lambda_0}(N)$, none of the open clusters intersecting $B((2\alpha + K + 2)N)$ are frozen.*

By Lemma 5.2.27 we choose $\lambda_0 = \lambda_0(\alpha, \varepsilon, K)$ and $N_0 = N_0(\alpha, \varepsilon, K)$ such that the event

$$E_0 := \mathcal{N}_c(\lambda_0, 1/24, 2\alpha + K + 4, N)$$

has probability at least $1 - \varepsilon/20$ for $N \geq N_0$. Then by Observation 5.3.2 we have that none of open clusters intersecting $B((2\alpha + K + 2)N)$ are frozen. In particular, if a vertex $v \in B((2\alpha + K + 2)N)$ is closed at time $p_\lambda(N)$, then it is $p_{\lambda_0}(N)$-closed. Moreover, if $v \in B((2\alpha + K + 2)N)$ is open at time $p_\lambda(N)$, then it is $p_\lambda(N)$-open. This finishes Step 1.

Let $\theta \in \left(0, \frac{1 \wedge \alpha}{2}\right)$. For $i = 1, 2$, let $BA^i = BA^i(\theta)$ denote the set of vertices $v \in B(KN)$ such that there are $\tilde{x}(v) = (\tilde{x}_1(v), \tilde{x}_2(v))$, $\tilde{y}(v) =$

$(\tilde{y}_1(v), \tilde{y}_2(v)) \in \mathcal{C}_a(v;\lambda)$ such that

$$\begin{aligned}\tilde{y}_i(v) - \tilde{x}_i(v) &= d(\tilde{x}(v), \tilde{y}(v)) \\ &= \operatorname{diam}(\mathcal{C}_a(v;\lambda)) \in ((\alpha-\theta)N, (\alpha+\theta)N).\end{aligned} \tag{5.5.13}$$

Note that

$$\begin{aligned}\{\exists v \in B(KN) \text{ s.t. } \operatorname{diam}(\mathcal{C}_a(v;\lambda)) \in ((\alpha-\theta)N, (\alpha+\theta)N)\} \\ = \left\{BA^1 \cup BA^2 \neq \emptyset\right\}.\end{aligned} \tag{5.5.14}$$

Let $u \in BA^2$. In the following we define quantities which depend on the value of u. In notation we indicate the dependence on u in the first appearance of these quantities, or when we want to emphasize this dependence. For each $u \in BA^2$ we fix a pair $(\tilde{x}, \tilde{y}) = (\tilde{x}, \tilde{y})(u)$ which satisfies (5.5.13). It can happen that there are more than one candidates for $\tilde{x}$ or $\tilde{y}$. In this case we choose one of them in some deterministic way. (E.g we can set $\tilde{x}$ $(\tilde{y})$ as the leftmost vertex among the candidates.) Let $x = x(u)$ $(y(u))$ denote a neighbour of $\tilde{x}$ $(\tilde{y})$ below $\tilde{x}$ (above $\tilde{y}$). The active cluster $\mathcal{C}_a(u;\lambda)$ lies between the horizontal lines passing through x and y denoted by e_x and e_y. Since $\theta < \alpha/2$, the outer boundary of $\mathcal{C}_a(u;\lambda)$ provides two non-touching closed half plane arms in $x + \mathbb{Z} \boxtimes [0,\infty)$ to distance $\alpha N/2$ starting from x. Since $\partial\mathcal{C}_a(u;\lambda) \subset B((2\alpha+K+2)N)$, by Step 1, on the event E_0 these arms are $p_{\lambda_0}(N)$-closed. We denote the one on the left (right) hand side by $c_L = c_L(u)$ $(c_R = c_r(u))$. Apart from their common starting point, c_L and c_R do not even touch, since any active path connecting $\tilde{x}$ to $\tilde{y}$ separates them. Since x is a closed frozen vertex, there is at least one open frozen neighbour of x. From this vertex there is a $p_\lambda(N)$-open arm $o_B = o_B(u)$ to distance at least $N/2$. See Figure 5.3 for more details. Let $\beta, \beta' \in (0,1)$ with $\beta < \beta'$. Recall the definition of the events $\mathcal{NA}(\beta,\beta') := \mathcal{NA}(\beta,\beta',\lambda,\lambda_0,2\alpha+K+2,N)$ and $\mathcal{NC}(\beta,\beta') := \mathcal{NC}(\beta,\beta',\lambda,\lambda_0,2\alpha+K+2,N)$ from Corollary 5.2.13 and 5.2.22. In the following we introduce the constants $\alpha_i > 0$ for $i = 1,2,3$ such that $\alpha_i/\alpha_{i+1} \gg 1$. Let $\alpha_3 \in \left(0, \frac{\alpha\wedge 1}{2}\right)$. Let $z = z(u) \in V$ such that $x = x(u) \in [-\alpha_3 N, \alpha_3 N] \boxtimes (-\alpha_3 N, \alpha_3 N] + \lfloor \alpha_3 N \rfloor z$. Note that $z \in B\left(\left\lceil \frac{2\alpha+K+2}{\alpha_3} \right\rceil\right)$. We define $B_3 = B_3(u) := B(\lfloor \alpha_3 N \rfloor z; \alpha_3 N)$. Note that throughout the arguments below, we will assume that $\alpha_1 > \alpha_2 > \alpha_3$, however, we will set their precise values only in later stages of the proof.

Step 2. *We show that with probability close to* 1, *there is only one frozen cluster close to* $x = x(u)$ *for all* $u \in BA^2$.

Let $\alpha_1 \in \left(0, \frac{\alpha\wedge 1}{2}\right)$, $B_1 = B_1(u) := B(\lfloor \alpha_3 N \rfloor z; \alpha_1 N)$ and $A_1 = A_1(u) := A\left(\lfloor \alpha_3 N \rfloor z; \alpha_1 N, \frac{\alpha\wedge 1}{2} N\right)$. Suppose that there are at least two different frozen clusters in B_1. On the event E_0 we find $5,2$ mixed near critical arms in A_1 : the two $p_{\lambda_0}(N)$-closed arms c_L and c_R, the two $p_\lambda(N)$-open arms from the two frozen clusters, and a $p_{\lambda_0}(N)$-closed arm separating them. Let $E_1 := \mathcal{NA}\left(\alpha_1, \frac{\alpha\wedge 1}{2}\right)$. Hence we get:

Claim 5.5.9. *On the event* $E_0 \cap E_1$, $\forall u \in BA^2$, *there is a unique frozen cluster denoted by* $F = F(u)$ *which intersects* $B_1(u)$. *Let* $\lambda_F = \lambda_F(u) \in [\lambda_0, \lambda]$ *such*

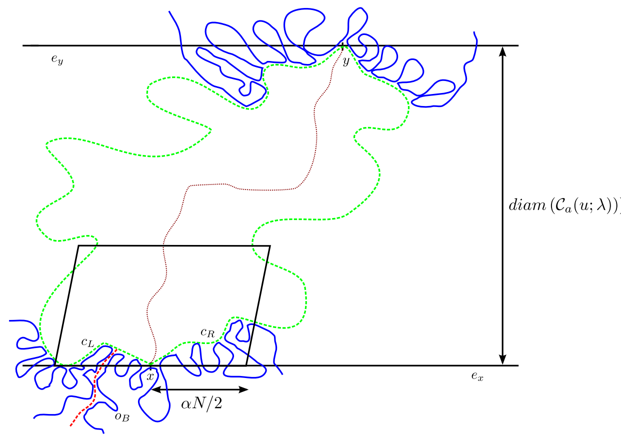

Figure 5.3: The closed boundary of $\mathcal{C}_a(\lambda)$ give rise to the closed arms c_L and c_R from x to $\partial B(x;\alpha N/2)$. The frozen vertex neighbouring x provides the arm o_B.

that F froze at $p_{\lambda_F}(N)$. On $E_0 \cap E_1$, a vertex in $B_1(u)$ is open in the N-parameter frozen percolation process at time $p_{\lambda_F}(N)$ if and only if it is $p_{\lambda_F}(N)$-open.

In the following two steps we write open (closed) for $p_{\lambda_F}(N)$-open ($p_{\lambda_F}(N)$-closed) if it is not stated otherwise. We finish Step 2 by applying Corollary 5.2.13 and we set α_1 such that

$$\mathbb{P}(E_1) \geq 1 - \varepsilon/20 \tag{5.5.15}$$

for $N \geq N_1(\varepsilon, \lambda_0, \lambda, \alpha, K)$.

Step 3. *We say that a circuit is $p_{\lambda_F}(N)$-open-closed, or simply open-closed, if it consists of a $p_{\lambda_F}(N)$-open and a $p_{\lambda_F}(N)$-closed arc. Suppose that there is a $p_{\lambda_F}(N)$-open-closed circuit close to and around x. We show that with probability close to 1, no matter how we change the τ values inside this circuit, the N-parameter frozen percolation process does not change till time $p_\lambda(N)$ outside of the circuit.*

Let $\alpha_2 \in \left(0, \alpha_1 \wedge \frac{1}{4}\right)$, and $\beta_2 \in (\alpha_2, \alpha_1)$ be some intermediate scale. We define

$$\begin{aligned}
B_2 = B_2(u) &:= B(\lfloor \alpha_3 N \rfloor z; \alpha_2 N), \\
B_2' = B_2'(u) &:= B(\lfloor \alpha_3 N \rfloor z; \beta_2 N), \\
A_2 = A_2(u) &:= A(\lfloor \alpha_3 N \rfloor z; \alpha_2 N, \alpha_1 N) \\
A_2' = A_2'(u) &:= A(\lfloor \alpha_3 N \rfloor z; \beta_2 N, \alpha_1 N).
\end{aligned}$$

Let $BL = BL(u)$ denote the set of bordering lines of $F \setminus B_2'$, that is the top- and bottom-most horizontal, left- and rightmost vertical lines which intersect

$F \setminus B_2'$. We rule out the case where there is a line in BL which intersects B_2' in the following technical claim.

Claim 5.5.10. *Let*

$$E_2' = \mathcal{NA}(2\beta_2, \alpha_1 - 2\beta_2) \cap \mathcal{NC}(2\beta_2, 2\alpha_1). \tag{5.5.16}$$

Then

$$E_0 \cap E_1 \cap E_2' \subset E_0 \cap E_1 \cap \left\{\forall u \in BA^2, \forall e \in BL(u) \text{ we have } e \cap (F \setminus B_2') = \emptyset\right\}.$$

Proof of Claim 5.5.10. Let $u \in BA^2$. When the bottom-most line of $F \setminus B_2'$ intersects B_2', then $F \subseteq (\mathbb{Z} \boxtimes [-\beta_2 N, \infty)) + \lfloor \alpha_3 N \rfloor z$. We see 4 half plane arms: c_L, c_R give two closed and o_B gives an open arm, a fourth closed half plane arm separates F from the line $\mathbb{Z} \boxtimes \{\lfloor \beta_2 N \rfloor\} + \lfloor \alpha_3 N \rfloor z$. Hence $\mathcal{NA}^c(2\beta_2, \alpha_1 - 2\beta_2)$ occurs.

If the topmost line of $F \setminus B_2'$ intersects B_2', then the closed arms c_L and c_R stay in the parallelogram

$$[-\alpha_1 N, \alpha_1 N] \boxtimes [-\beta_2 N, \beta_2 N] + \lfloor \alpha_3 N \rfloor z.$$

Hence we get a closed horizontal crossing of it, that is the event $\mathcal{NC}^c(2\beta_2, 2\alpha_1)$ occurs.

When a leftmost bordering line of $F \setminus B_2'$ intersects B_2', then we find that the arms in $A(\lfloor \alpha_3 N \rfloor z; \beta_2 N, \alpha_1 N)$ induced by c_L, c_R and o_B stay in half plane

$$[-2\beta_2 N, \infty) \times \mathbb{R} + \lfloor \alpha_3 N \rfloor z. \tag{5.5.17}$$

The frozen cluster F is separated from the line $\{-2\beta_2 N\} \times \mathbb{R} + \lfloor \alpha_3 N \rfloor z$. This provides an additional closed arm in the half plane (5.5.17), which together the arms induced by c_L, c_R and o_B give 4 half plane arms, hence the event $\mathcal{NA}^c(\beta_2, \alpha_1 - 2\beta_2)$ occurs.

The case when the rightmost bordering line of $F \setminus B_2'$ intersects B_2' can be treated similarly.

With the notation (5.5.16) we get that on the event $E_0 \cap E_1 \cap E_2'$, none of the lines of BL intersect B_2', which finishes the proof of Claim 5.5.10. □

Now we proceed with Step 3. Let $u \in BA^2$. Suppose that there is an open-closed circuit $OC = OC(u)$ around x in B_2. Let $R = R(u)$ denote the union of the finite connected components of $\mathbb{T} \setminus OC$. Let us change the τ values of the vertices in R in some arbitrary non-degenerate way (that is, the new τ values are all different), but keep the original values outside R. Let us run the N-parameter frozen percolation dynamics for this modified set of τ values.

First we consider the case where this new process differs from the old one outside of R at some time $t \in [0, p_{\lambda_F}(N)]$. The closed arc of OC stays closed till time $p_{\lambda_F}(N)$ in both processes. Hence it acts as a barrier for the effect of τ values in R. The open arc of OC is a subset of F. By Claim 5.5.10 on the event $E_0 \cap E_1 \cap E_2'$ if these two processes differ outside R, then in the new one a frozen cluster F' emerged before time $p_{\lambda_F}(N)$. Since the two processes differ outside R, we have $F' \setminus R \neq F \setminus R$. On the event E_0 there are no $p_{\lambda_0}(N)$-open clusters with diameter at least $N/4$ intersecting $B((2\alpha + K + 2)N)$. Since $\alpha_2 < 1/4$ we get that F' froze in the new process at time $p_{\lambda_{F'}}(N)$ with $\lambda_{F'} \in [\lambda_0, \lambda_F]$. Let BL' denote the set of bordering lines of $F' \setminus B_2'$. Since $\lambda_{F'} \in [\lambda_0, \lambda_F]$, the arguments of the proof of Claim 5.5.10 applied to the new process give that on the event $E_0 \cap E_1 \cap E_2'$ none of the lines of BL' intersect B_2'. Hence $F' \setminus R$ has two connected components F_1' and F_2' such that $\operatorname{diam}(F_i') < N$ for $i \in \{1,2\}$, but $\operatorname{diam}(F_1' \cup F_2') \geq N$. Since $R \subset B_2$, each of F_1', F_2' contains a $p_{\lambda_{F'}}(N)$-open arm in the annulus $A_2'' = A_2''(u) := A(\lfloor \alpha_3 N \rfloor z; \alpha_2 N, \beta_2 N)$. When for some $i \in \{1,2\}$ F_i' lies above c_L and c_R, then we get a $4,3$ near critical arm event: the closed arms induced by c_L, c_R and the open arm induced by F_i' stay above e_x, and o_B provides the fourth arm in A_2''. Hence $\mathcal{NA}^c(\alpha_2, \beta_2)$ occurs. If both of F_1', F_2' lie below c_L and c_R then we get a $5,2$ near critical mixed arm event in A_2'' : c_L, c_R induce closed half plane arms in A_2''. F_1', F_2' induce two open arms. Since F_1' and F_2' are different connected components of $F' \setminus R$, there is a fifth, $p_{\lambda_{F'}}(N)$-closed, arm separating F_1' and F_2' in A_2''. Hence $\mathcal{NA}^c(\alpha_2, \beta_2)$ occurs. Let $E_2 = E_2' \cap \mathcal{NA}(\alpha_2, \beta_2)$.

Let us consider the other case where the new and the old process differ from each other outside of R at some time $t \in (p_{\lambda_F}(N), p_\lambda(N)]$, but they agree outside of R till time $p_{\lambda_F}(N)$. Since till time $p_{\lambda_F}(N)$ the two processes coincide outside of R, then a frozen cluster F' is formed at time $p_{\lambda_F}(N)$ in the new process. Moreover, $F' \setminus R = F \setminus R$. However, the two processes differ at some time $t \in (p_{\lambda_F}(N), p_\lambda(N)]$, hence an additional frozen cluster F'' has to emerge in this time period using some of the vertices in R. This induces the $5,2$ near critical mixed arm event of Step 2. Hence we proved the following claim.

Claim 5.5.11. *On the event $E_0 \cap E_1 \cap E_2$, we have that $\forall u \in BA^2$, if there is a $p_{\lambda_F}(N)$-open-closed circuit around $x = x(u)$ in $B_2(u)$ then no matter how we change the τ values inside this circuit, the frozen percolation process outside it does not change till time $p_\lambda(N)$.*

We finish Step 3 by applying Corollary 5.2.13 and 5.2.24: we fix the value of β_2 and α_2 such that

$$\mathbb{P}(E_2) \geq 1 - \varepsilon/20 \tag{5.5.18}$$

for $N \geq N_2(\varepsilon, \lambda_0, \lambda, \alpha, K)$.

Step 4. *We show that with probability close to* 1, *there is a $p_{\lambda_F}(N)$-open-closed circuit around x, such that the location where its colour changes in the circuit is 'far' above x.*

Let $u \in BA^2$. Let $\alpha_3 \in (0, \alpha_2)$, $B_3 = B_3(u) := B(\lfloor \alpha_3 N \rfloor z; \alpha_3 N)$ and $A_3 = A_3(u) := A(\lfloor \alpha_3 N \rfloor z; \alpha_3 N, \alpha_2 N)$. Let $\delta_3 \in (\alpha_3, \alpha_2)$ be an intermediate scale. We cut the annulus A_3 into three subannuli using two other intermediate scales β_3, β_3' with $\alpha_3 < \delta_3 < \beta_3 < \beta_3' < \alpha_2$:

$$A_{3,0} = A_{3,0}(u) := A(\lfloor \alpha_3 N \rfloor z; \alpha_3 N, \beta_3 N),$$

$$A_{3,1} = A_{3,1}(u) := A(\lfloor \alpha_3 N \rfloor z; \beta_3 N, \beta_3' N),$$
$$A_{3,2} = A_{3,2}(u) := A(\lfloor \alpha_3 N \rfloor z; \beta_3' N, \alpha_2 N).$$

Let $\bar{c}_L$ ($\bar{c}_R$) denote the closed arm induced by c_L (c_R) in $A_{3,1}$.

If c_L and c_R are not connected by a closed path in $A_{3,0} \cap \mathcal{C}_a(u;\lambda)$, then there is a open arm separating them. Hence we see a near critical $4,3$ arm event: c_L, c_R and the separating open arm induce disjoint half plane arms in $A_{3,0}$, and the fourth arm in $A_{3,0}$ is induced by o_B. Thus the event $\mathcal{NA}^c(\alpha_3, \beta_3)$ occurs.

If $\bar{c}_L \subseteq [-\beta_3' N, -\beta_3 N] \boxtimes [-\alpha_3 N, \delta_3 N]$ or $\bar{c}_R \in [\beta_3 N, \beta_3' N] \boxtimes [-\alpha_3 N, \delta_3 N]$, then we find a closed horizontal crossing in a narrow parallelogram. Hence the event $\mathcal{NC}^c(\alpha_3 + \delta_3, \beta_3' - \beta_3)$ occurs.

In the following we assume that both $\bar{c}_L$ and $\bar{c}_R$ leave the corresponding parallelograms. Let w_L (w_R) be an open frozen vertex neighbouring a vertex of $\bar{c}_L$ ($\bar{c}_R$) which is outside of the aforementioned parallelogram.

Suppose that there is no open arc in A_3 connecting w_L to o_B. Since w_L is open frozen at time $p_{\lambda_F}(N)$, it has a $p_{\lambda_F}(N)$-open path to distance $N/2$. Let o_L denote the part of this path till the first time it exits A_3. Note that o_L and o_B are disjoint, and they are not connected by an open path inside A_3. We have two cases depending on where o_L leaves A_3.

When it leaves A_3 by exiting its outer parallelogram, than we get a $5,2$ near critical arm event in $A_{3,2}$: two half plane closed arms induced by c_L and c_R, two open arms induced by o_L and o_B an extra closed arm separates o_L and o_B in $A_{3,2}$. Hence the event $\mathcal{NA}^c(\beta_3', \alpha_2)$ occurs. See Figure 5.4.

When o_L leaves A_3 by entering its inner parallelogram, then we get a similar $5,2$ arm event in $A_{3,0}$. Thus $\mathcal{NA}^c(\alpha_3, \beta_3)$ happens. In a similar way we can show that when w_R is not connected to o_B in $A_{3,x}$, then $\mathcal{NA}^c(\beta_3', \alpha_2) \cup \mathcal{NA}^c(\alpha_3, \beta_3)$ occurs.

See Figure 5.5. Note that $w_L, w_R \in (\mathbb{Z} \boxtimes [\delta_2 N, \alpha_2 N]) + \lfloor \alpha_3 N \rfloor z$. With the notation

$$E_3 = \mathcal{NC}(\alpha_3 + \delta_3, \beta_3' - \beta_3) \cap \mathcal{NA}(\alpha_3, \beta_3) \cap \mathcal{NA}(\beta_3', \alpha_2) \cap \mathcal{NA}(\alpha_3, \beta_3)$$

we proved the following claim.

Claim 5.5.12. *We have that on the event $E_0 \cap E_1 \cap E_2 \cap E_3$, $\forall u \in BA^2$ there is a $p_{\lambda_F(u)}(N)$-open-closed circuit in $A_3(u)$ such that locations where the colour changes in the circuit is $(\mathbb{Z} \boxtimes [\delta_3 N, \alpha_2 N]) + \lfloor \alpha_3 N \rfloor z$.*

We finish Step 4 by choosing the values of β_3, β_3' and δ_3. The probability of E_3 is an increasing function of α_3 for $\beta_3, \beta_3', \delta_3$ fixed. By Corollary 5.2.13 and 5.2.24 we choose the value of $\beta_3, \beta_3', \delta_3, \alpha_3$ such that the probability of the event E_3 is at least $1 - \varepsilon/20$. We only fix $\beta_3, \beta_3', \delta_3$ and require α_3 to be small but unspecified so that

$$\mathbb{P}(E_3) \geq 1 - \varepsilon/20 \tag{5.5.19}$$

for $N \geq N_3(\varepsilon, \alpha_3, \lambda_0, \lambda, \alpha, K)$. We choose the value of α_3 in Step 6.

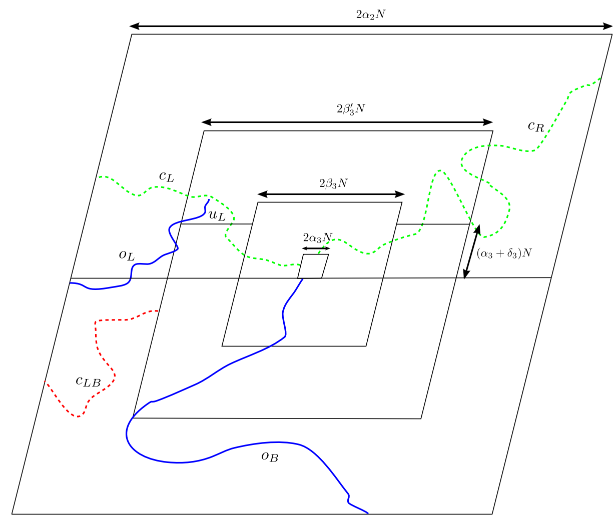

Figure 5.4: The closed arm c_{LB} separates o_L and o_B in $A_{3,2,x}$. Hence $c_L, c_R, o_B, c_{LB}, o_L$ give 5, 2 near critical mixed arms.

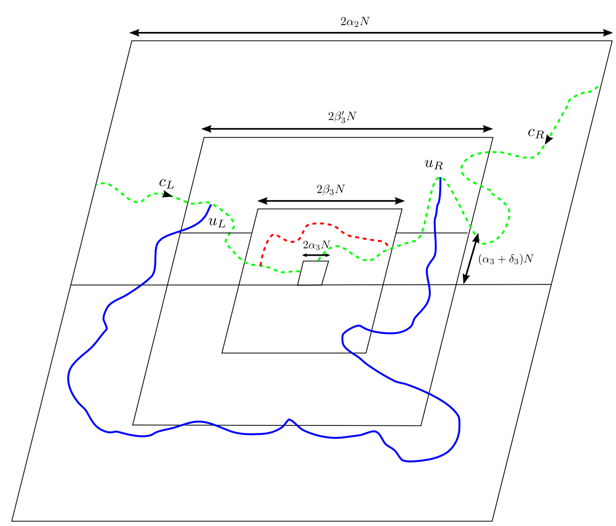

Figure 5.5: The circuit around $B_{3,x}$ consists of the open arc drawn with continuous line, subpaths of c_L and c_R and the closed arc in $A_{3,0,x}$.

Before Step 5, let us summarize what we have proved up to now. Let $u \in BA^2$, and suppose that the event $E_0 \cap E_1 \cap E_2 \cap E_3(\alpha_3)$ holds. Consider the circuit OC which we constructed in Step 4. It has a special property: as we walk from the outside of $B_2 = B(\lfloor \alpha_3 N \rfloor z; \alpha_2 N)$ on any of the closed arms c_L or c_R towards x, we hit the closed part of OC at its endpoints for the first time. See Figure 5.5 for more details. Let $\mathcal{OC}$ denote the outermost open-closed circuit in A_3 satisfying the conditions appearing in Claim 5.5.12 and this special property. Let $\mathcal{R}$ denote the union of finite connected components of $\mathbb{T} \setminus \mathcal{OC}$. Let r_o and r_c denote the open and closed parts of $\mathcal{OC} = \partial\mathcal{R}$. The pair $(\mathcal{R}, r_c)$ and the configuration in $\mathbb{T}\backslash\mathcal{R}$ satisfies the following conditions:

1. $\mathcal{R}$ is an induced subgraph of $\mathbb{T}$ such that $cl(\mathcal{R})$ is connected, (by Claim 5.5.12)
2. $B(\lfloor \alpha_3 N \rfloor z; \alpha_3 N) = B_3 \subseteq \mathcal{R} \subseteq B_2 = B(\lfloor \alpha_3 N \rfloor z; \alpha_2 N)$ (by Claim 5.5.12)
3. $\partial\mathcal{R}$ is disjoint union of non-empty self avoiding paths r_c and r_o, which are oriented such that $\mathcal{R}$ lies on the right when we walk along them, (by Claim 5.5.12)
4. $r_c \subseteq [-\alpha_2 N, \alpha_2 N] \boxtimes [-\alpha_3 N, \alpha_2 N] + \lfloor \alpha_3 N \rfloor z$, (by the proof of Claim 5.5.12)
5. the endpoints of r_c denoted by s_L and s_R lie in the parallelogram
$$[-\alpha_2 N, \alpha_2 N] \boxtimes [\delta_3 N, \alpha_3 N] + \lfloor \alpha_3 N \rfloor z,$$
(by Claim 5.5.12)
6. when we walk along c_L (c_R) towards x, we hit $\partial\mathcal{R}$ first at vertex s_L (s_R), (by the proof of Claim 5.5.12)
7. for every vertex $v \in r_o$, there is a closed path in $B_2 \setminus \mathcal{R}$ to ∂B_2, ($\mathcal{OC}$ is outermost)
8. for every vertex $v \in r_c$, there is an open path in $B_2 \setminus \mathcal{R}$ to ∂B_2 or to $(c_L \cup c_R) \setminus cl(\mathcal{R})$. ($\mathcal{OC}$ is outermost)

Note that the first three conditions coincide with the first three conditions for the pair $(\mathcal{R} - \lfloor \alpha_3 N \rfloor z, r_c - \lfloor \alpha_3 N \rfloor z)$ being $(\alpha_3 N, \alpha_2 N)$-outer-regular of Definition 5.5.5. We add an extra condition in the next step.

Note that the vertex x has two non-touching closed arms to r_c. Moreover, by Condition 6, x is one of the lowest vertices in $\mathcal{R}$ with this property. With the notation of Definition 5.5.1 we have that $x \in \mathcal{L}(\mathcal{R}, r_c)$ in the N-parameter frozen percolation process at time $p_{\lambda_F}(N)$.

Step 5. Let $u \in BA^2$. *Suppose that the event $E_0 \cap E_1 \cap E_2 \cap E_3$ holds.* Let $\mathcal{W} = \mathcal{W}(u)$ denote the set of the connected components of

$$\mathcal{R} \cap (\mathbb{Z} \boxtimes [-\lfloor \alpha_3 N \rfloor + 1, \lfloor 5\alpha_3 N \rfloor - 1]).$$

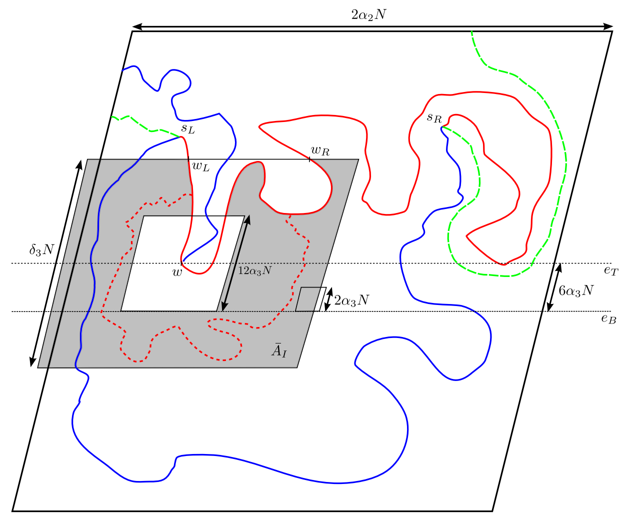

Figure 5.6: The grey area represents $\bar{A}_I$. If there is no open arm in $\bar{A}_I$ then there is a closed arc in $\bar{A}_I$. This contradicts with x being one of the lowest vertices of $\mathcal{C}_a(\lambda)$.

Let $S_{mid}(\mathcal{R})$ denote the unique element of $\mathcal{W}$ which contains B_3 as a subset. We show that with probability close to 1, $\partial S_M \cap r_c = \emptyset$.

We define $e_T = e_T(u) := (\mathbb{Z} \boxtimes \{\lfloor 5\alpha_3 N \rfloor + 1\}) + \lfloor \alpha_3 N \rfloor z$ and $e_B = e_B(u) := (\mathbb{Z} \boxtimes \{-\lfloor \alpha_3 N \rfloor - 1\}) + \lfloor \alpha_3 N \rfloor z$. Suppose that $\partial S_M \cap r_c \neq \emptyset$, let $w \in \partial S_{mid}(\mathcal{R}) \cap r_c \cap e_T$. Consider the parallelogram $\bar{B} = B(w; \delta_3 N/2)$. Let w_L and w_R denote the vertices of r_c where we exit $\bar{B}$ the first time as we walk on r_c starting from w towards s_L and s_R. The part of r_c between w_L and w_R cuts $\bar{B}$ into two pieces. Let $\bar{B}_I$ ($\bar{B}_E$) denote the part which is on the right (left) hand side of r_c when we walk from w_L to w_R. Let $\bar{A}_I = \bar{B}_I \setminus B(w; 6\alpha_3 N)$ and $\bar{A}_E = \bar{B}_E \setminus B(w; 6\alpha_3 N)$. By Condition 8 above $\bar{A}_E$ contains an open arm. We claim that $\bar{A}_I$ also contains an open arm. Suppose the contrary. Then there must be a closed non self-touching arc in $\bar{A}_I$ preventing the occurrence of the open arm. Note that this arc is contained in $\mathcal{R}$. Then the lowest vertex of this arc has two disjoint $p_{\lambda_F}(N)$-closed arms to r_c, and it lies lower than $x \in B := B(\lfloor \alpha_3 N \rfloor z; \alpha_3 N)$. This contradicts $x \in \mathcal{L}(\mathcal{R}, r_c)$ which was shown in the lines before Step 4. See Figure 5.6. Hence $\bar{A}_I$ has an open arm, which together with the open arm of $\bar{A}_E$ and the two closed arms of w provide a 4, 3 near critical mixed arm event. Hence the event $E_4^c = \mathcal{NA}^c(6\alpha_3, \delta_3/2)$ occurs. Thus we arrive to the following claim and we finish Step 5.

Claim 5.5.13. *On the event* $E_0 \cap E_1 \cap E_2 \cap E_3 \cap E_4$, *we have* $\partial S_{mid}(\mathcal{R}) \cap r_c = \emptyset$.

Step 6. *Recall Definition 5.5.5. We show that with probability close to* 1, *we can cut down some parts of* $\mathcal{R}$ *and get a pair* $\tilde{\mathcal{R}}$ *and* $\tilde{r}_c$ *such that the pair* $\left(\tilde{\mathcal{R}} - \lfloor \alpha_3 N \rfloor z, \tilde{r}_c - \lfloor \alpha_3 N \rfloor z\right)$ *is* $(\alpha_3 N, \alpha_2 N)$*-regular and*

$$\mathcal{L}(\mathcal{R}, r_c) \cap B = \mathcal{L}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right) \cap B.$$

Let $u \in BA^2$. Suppose that the event $E_0 \cap E_1 \cap E_2 \cap E_3 \cap E_4$ occurs. Let $\tilde{\mathcal{R}} = \tilde{\mathcal{R}}(u)$ be the connected component of $S_{mid}(\mathcal{R})$ in $\mathcal{R} \setminus \bigcup_{S \in \mathcal{W}:\, \partial S \cap r_c \neq \emptyset} cl(S)$ and $\tilde{r}_c = \partial \tilde{\mathcal{R}} \setminus r_o$. The conditions before Step 5 and Claim 5.5.13 gives that the pair $\left(\tilde{\mathcal{R}} - \lfloor \alpha_3 N \rfloor z, \tilde{r}_c - \lfloor \alpha_3 N \rfloor z\right)$ is $(\alpha_3 N, \alpha_2 N)$-regular.

For $R \subset \mathbb{T}$ and $r \subset \partial R$ let $\mathcal{TA}(R, r)$ denote the set of closed vertices $v \in R$ such that v has two non-touching closed arms in R to r. Let M denote the connected component of $S_{mid}(R)$ in $R \setminus e_T$. We show the following:

Claim 5.5.14. *Let*

$$E_5 := \mathcal{NA}(6\alpha_3, \beta_4) \cup \mathcal{NA}(\beta_4, \delta_3/2). \tag{5.5.20}$$

On the event $\bigcap_{i=0}^{5} E_i \ \forall u \in BA^2$*, the pair* $\left(\tilde{\mathcal{R}} - \lfloor \alpha_3 N \rfloor z, \tilde{r}_c - \lfloor \alpha_3 N \rfloor z\right)$ *is* $(\alpha_3 N, \alpha_2 N)$*-regular, and*

$$\mathcal{TA}(\mathcal{R}, r_c) \cap M = \mathcal{TA}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right) \cap M.$$

In particular,

$$\mathcal{L}(\mathcal{R}, r_c) \cap B = \mathcal{L}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right) \cap B.$$

Proof of Claim 5.5.14. From the definition of $\left(\tilde{\mathcal{R}}, \tilde{r}_c\right)$ it follows that

$$(\mathcal{TA}(\mathcal{R}, r_c) \cap M) \subset \left(\mathcal{TA}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right) \cap M\right).$$

Hence it is enough to show that $\left(\mathcal{TA}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right) \setminus \mathcal{TA}(\mathcal{R}, r_c)\right) \cap M = \emptyset$. Suppose the contrary, that is $\exists v \in \left(\mathcal{TA}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right) \setminus \mathcal{TA}(\mathcal{R}, r_c)\right) \cap M$. Let c_v^1 and c_v^2 denote two non-touching closed arms starting from v and ending at $v^1 \in \tilde{r}_c$ and $v^2 \in \tilde{r}_c$ respectively. Since $v \in \mathcal{TA}(\mathcal{R}, r_c) \setminus \mathcal{TA}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right)$, we can assume that c_v^1 cannot be extended in such a way that it connects to r_c and this extension is disjoint from and does not touch c_v^2. Hence $v^1 \in \tilde{r}_c \setminus r_c$, and $v^1 \in e_T$. Let $S \in \mathcal{W}$ such that $v^1 \in \partial S$. Note that $\partial S \cap r_c \neq \emptyset$. Let s^1, s^2 denote the endpoints of the connected component of v^1 in $\partial S \cap e_T$. At least one of s^1 and s^2 is in r_c. Let $s^1 \in r_c$. Let $\beta_4 \in (6\alpha_3, \delta_3/2)$ be an intermediate scale. We divide the annulus $A\left(v^1; 6\alpha_3 N, \delta_3 N/2\right)$ into the annuli

$$A_{4,0} = A\left(v^1; 6\alpha_3 N, \beta_4 N\right),$$

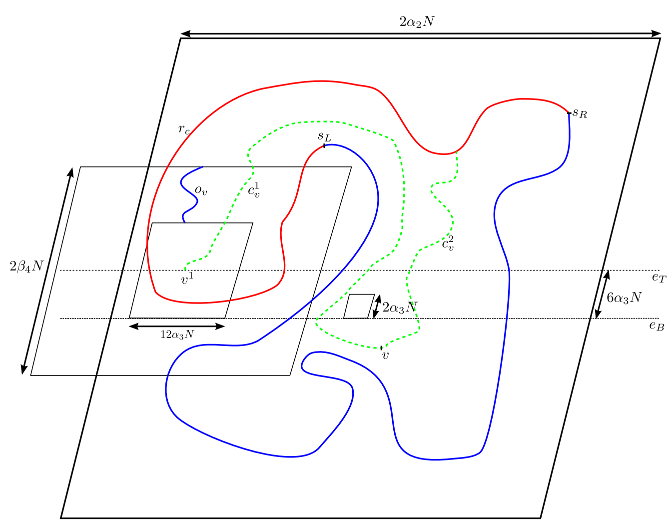

Figure 5.7: If $c_v^2 \cap A_{4,0} = \emptyset$, we see 4 half plane arms in $A_{4,0}$: the two closed induced by r_c, a closed arm c_v^1, and an open arm o_v which separates c_v^1 from r_c.

$$A_{4,1} = A\left(v^1; \beta_4 N, \delta_3 N/2\right).$$

We have two cases. If $c_v^2 \cap A_{4,0} \neq \emptyset$, then we see 4 half plane arms in $A_{4,1}$: r_c provides two closed arms, and each of c_v^1 and c_v^2 gives one closed arm. Hence the event $\mathcal{NA}^c\left(6\alpha_3, \beta_4\right)$ occurs. If $c_v^2 \cap A_{4,0} = \emptyset$, we have 4 half plane arms in $A_{4,0}$: r_c provides two closed arms, c_v^1 another closed arm, moreover, we get an open arm which separates c_v^1 from r_c. See Figure 5.7 for more details. Hence the event $\mathcal{NA}^c\left(\beta_4, \delta_3/2\right)$ occurs. By (5.5.20) this finishes the proof of Claim 5.5.14. □

By Corollary 5.2.13 we set α_3 such that

$$\mathbb{P}\left(E_4 \cap E_5\right) \geq 1 - \varepsilon/20 \tag{5.5.21}$$

for $N \geq N_5\left(\varepsilon, \alpha_3, \lambda_0, \lambda, \alpha, K\right)$. Let

$$E = E_0 \cap E_1 \cap E_2 \cap E_3 \cap E_4 \cap E_5.$$

The combination of the lines in the beginning of Step 1, (5.5.15), (5.5.18), (5.5.19) and (5.5.21) gives that

$$\mathbb{P}\left(E\right) \geq 1 - \varepsilon/4 \tag{5.5.22}$$

for $N \geq \bigvee_{i=0}^{5} N_i$. This finishes Step 6.

Step 7. *We set $\theta > 0$ such that $\mathbb{P}_N\left(BA^2 \neq \emptyset\right) < \varepsilon/2$ for large N, and conclude the proof of Proposition 5.3.5.*

For $v \in V$, let

$$Z(v) := \left\{ \exists u \in BA^2 \text{ such that } z(u) = v \right\}.$$

Hence

$$\left\{ BA^2 \neq \emptyset \right\} = \bigcup_{v \in B\left(\left\lceil \frac{2\alpha + K + 2}{\alpha_3} \right\rceil \right)} Z(v)$$

and

$$\mathbb{P}_N \left(BA^2 \neq \emptyset, E \right) \leq \sum_{v \in B\left(\left\lceil \frac{2\alpha + K + 2}{\alpha_3} \right\rceil \right)} \mathbb{P}_N \left(Z(v) \cap E \right) \tag{5.5.23}$$

Note that on the event $Z(v) \cap E$, Claim 5.5.9 and the arguments above give that $\mathcal{C}_a(u;\lambda), F(u), \lambda_F(u), \mathcal{R}(u), r_c(u), \tilde{\mathcal{R}}(u)$ and $\tilde{r}_c(u)$ do not depend on the choice of $u \in BA^2$ as long as $z(u) = v$. Except for $\mathcal{C}_a(u, \lambda)$, we omit the argument u from the notation above.

We set $k := \lfloor 1/2\theta \rfloor$. Recall that $d(x, y) = d(\tilde{x}, \tilde{y}) + \sqrt{3} = \operatorname{diam}(\mathcal{C}_a(u;\lambda)) + \sqrt{3}$, and $\operatorname{diam}(\mathcal{C}_a(u;\lambda)) \in ((\alpha - \theta) N, (\alpha + \theta) N)$. On the event $Z(v)$ there is a unique $l = l(y) \in [0, k-1] \cap \mathbb{Z}$ such that $x \in B_{l,k}$ where

$$\begin{aligned} B_{l,k} &= B_{l,k}(v) \\ &:= [-\alpha_3 N, \alpha_3 N] \boxtimes \left(\left(2\frac{l}{k} - 1 \right) \alpha_3 N, \left(2\frac{l+1}{k} - 1 \right) \alpha_3 N \right] + \lfloor \alpha_3 N \rfloor v. \end{aligned}$$

Recall from the lines above Step 5 we have $x \in \mathcal{L}(\mathcal{R}, r_c)$. From Claim 5.5.14 we have $\mathcal{L}(\mathcal{R}, r_c) \cap B = \mathcal{L}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right) \cap B$ where $B = B(\lfloor \alpha_3 N \rfloor v; \alpha_3 N)$. Hence on the event $Z(v) \cap E$, we have $\mathcal{L}\left(\tilde{\mathcal{R}}, \tilde{r}_c\right) \cap B_{l,k} \neq \emptyset$. Let (R, r) be a fixed pair. Hence

$$\begin{aligned} &\mathbb{P}_N \left(Z(v), E, (\mathcal{R}, r_c) = (R, r) \right) \\ &\quad = \mathbb{P}_N \left(Z(v), E, (\mathcal{R}, r_c) = (R, r), \mathcal{L}\left(\tilde{R}, \tilde{r}\right) \cap B_{l,k} \neq \emptyset \text{ at time } p_{\lambda_F}(N) \right) \end{aligned} \tag{5.5.24}$$

where $\left(\tilde{R}, \tilde{r}\right)$ denotes the pair we get when we cut down some parts of R as in Step 6.

For $t \in [0,1]$ and $J \subset V$ let

$$\mathcal{F}_t(J) := \sigma\left(\{\tau_w < s\} \,|\, w \in J, \, s \in [0,1] \right)$$

denote the σ-algebra generated by the τ values of the vertices in J up to time t.

Lemma 5.5.8 gives that the N-parameter frozen percolation process is adapted to the filtration $\left(\mathcal{F}_t(V)\right)_{t \in [0,1]}$. Hence for all $u \in BA^2$, $\{(\mathcal{R}, r_c) = (R, r)\} \in \mathcal{F}_{p_\lambda(N)}(V)$. Moreover, l and λ_F are $\mathcal{F}_{p_\lambda(N)}(V)$ measurable functions. By Claim 5.5.11 we have that on the event $Z(v) \cap E \cap \{(\mathcal{R}, r_c) = (R, r)\}$ the τ-values in R

do not influence the frozen percolation process in $V \setminus R$ up to time $p_{\lambda}(N)$. The arguments above combined with the fact that $\partial \mathcal{R}$ is a certain outermost circuit, we get that there is a function f such that $f\left(R, \bar{l}, \bar{\lambda}_F\right)$ is a $\mathcal{F}_{p_{\lambda}(N)}(V \setminus R)$-measurable function for all $R, \bar{l}, \bar{\lambda}_F$. Moreover, it satisfies

$$\begin{aligned}\mathbf{1}\left\{Z(v), E, (\mathcal{R}, r_c) = (R, r), l = \bar{l}, \lambda_F \in d\bar{\lambda}_F\right\} = \\ f\left(R, \bar{l}, \bar{\lambda}_F\right) \mathbf{1}\{Z(v), E\} \quad (5.5.25)\end{aligned}$$

with probability 1 for almost every $\bar{\lambda}_F$. Hence

$$\begin{aligned}&\mathbb{P}_N\left(\begin{matrix} Z(v), E, (\mathcal{R}, r_c) = (R, r), l = \bar{l}, \lambda_F \in d\bar{\lambda}_F, \\ \mathcal{L}\left(\tilde{R}, \tilde{r}\right) \cap B_{l,k} \neq \emptyset \text{ at time } p_{\lambda_F}(N) \end{matrix} \middle| \mathcal{F}_{p_{\lambda}(N)}(V \setminus R)\right) \\ &\quad = f\left(R, \bar{l}, \bar{\lambda}_F\right) \\ &\qquad \mathbb{P}_N\left(\begin{matrix} Z(v), E, \\ \mathcal{L}\left(\tilde{R}, \tilde{r}\right) \cap B_{\bar{l},k} \neq \emptyset \text{ at time } p_{\bar{\lambda}_F}(N) \end{matrix} \middle| \mathcal{F}_{p_{\lambda}(N)}(V \setminus R)\right), \quad (5.5.26)\end{aligned}$$

for $\bar{l} \in [0, k-1] \cap \mathbb{Z}$ and $\bar{\lambda}_F \in [\lambda_0, \lambda]$.

From Step 6 we have $\tilde{R} \subset R$. Claim 5.5.14 shows that we can apply Corollary 5.5.7 in the following. We have

$$\begin{aligned}&\mathbb{P}_N\left(Z(v), E, \mathcal{L}\left(\tilde{R}, \tilde{r}\right) \cap B_{\bar{l},k} \neq \emptyset \text{ at time } p_{\bar{\lambda}_F}(N) \,\middle|\, \mathcal{F}_{p_{\lambda}(N)}(V \setminus R)\right) \\ &\qquad \leq \mathbb{P}_N\left(\mathcal{L}\left(\tilde{R}, \tilde{r}\right) \cap B_{\bar{l},k} \neq \emptyset \text{ at time } p_{\bar{\lambda}_F}(N) \,\middle|\, \mathcal{F}_{p_{\lambda}(N)}(V \setminus R)\right) \\ &\qquad = \mathbb{P}_{p_{\bar{\lambda}_F}(N)}\left(\mathcal{L}\left(\tilde{R}, \tilde{r}\right) \cap B_{\bar{l},k} \neq \emptyset\right) \\ &\qquad \leq c_1 k^{-1} \quad (5.5.27)\end{aligned}$$

for $N \geq N_6(\lambda_0, \lambda, \alpha_3, \alpha_2, k)$ with $c_1 = c_1(\lambda_0, \lambda, \alpha_3, \alpha_2)$ of Corollary 5.5.7. A combination of (5.5.27) and (5.5.26) gives that

$$\begin{aligned}&\mathbb{P}_N\left(\begin{matrix} Z(v), E, (\mathcal{R}, r_c) = (R, r), l = \bar{l}, \lambda_F \in d\bar{\lambda}_F, \\ \mathcal{L}\left(\tilde{R}, \tilde{r}\right) \cap B_{l,k} \neq \emptyset \text{ at time } p_{\lambda_F}(N) \end{matrix} \middle| \mathcal{F}_{p_{\lambda}(N)}(V \setminus R)\right) \\ &\qquad \leq c_1 k^{-1} f\left(R, \bar{l}, \bar{\lambda}_F\right) \quad (5.5.28)\end{aligned}$$

for $N \geq N_6$. Hence

$$\mathbb{P}_N(Z(v) \cap E) \leq c_1 k^{-1}. \quad (5.5.29)$$

for $N \geq N_6$.

(5.5.29) combined with (5.5.23) we get that

$$\mathbb{P}_N\left(BA^2 \neq \emptyset, E\right) \leq \sum_{v \in B\left(\left\lceil \frac{2\alpha + K + 2}{\alpha_3} \right\rceil\right)} \mathbb{P}_N(Z(v) \cap E)$$

$$\leq c_2 k^{-1} \tag{5.5.30}$$

with $c_2 = c_2(\lambda_0, \lambda, \alpha_3, \alpha_2, K)$ for $N \geq N_6$. We set θ such that $k = \lfloor 1/2\theta \rfloor > 4c_2/\varepsilon$. A combination of (5.5.30) and (5.5.22) gives that

$$\begin{aligned}\mathbb{P}_N\left(BA^2 \neq \emptyset\right) &\leq \mathbb{P}_N\left(BA^2 \neq \emptyset,\, E\right) + \mathbb{P}_N\left(E^c\right) \\ &\leq c_2 k^{-1} + \varepsilon/4 < \varepsilon/2 \end{aligned}\tag{5.5.31}$$

for $N \geq N' = \bigvee_{i=0}^{6} N_i$.

A proof analogous to that of (5.5.31) gives that there is $N'' = N''(\alpha, \lambda, K)$

$$\mathbb{P}_N\left(BA^1 \neq \emptyset\right) < \varepsilon/2 \tag{5.5.32}$$

for $N \geq N''$. A combination of (5.5.14), (5.5.31) and (5.5.32) finishes the proof of Proposition 5.3.5.

□

5.A Appendix

5.A.1 Winding number of arms

Here we prove Proposition 5.2.7. The proof is motivated by [11]. There, among many other things, it was shown that when there are k disjoint open arms in $A(M, aM)$ $(a > 1)$, then, with conditional probability at least $1 - a^{-\varepsilon}$, and uniformly in M, are also k disjoint open arms which wind around the origin at least $c \log a$ times where c, ε are positive constants.

We prove a slightly different result, namely that if we have k disjoint arms with any colour sequence $\sigma \in \{o, c\}^k$ in $A(M, aM)$, than with conditional probability at least $1 - a^{-\varepsilon}$, these arms wind around the origin at in at least $c \log a$ disjoint subannuli of $A(a, b)$ for some $c, \varepsilon > 0$. Following [77], we recall the notion of well separated arms. We modify Definition 7 of [77] for annuli:

Definition 5.A.1. *Consider some annulus $A = A(v; M, \tau M)$ and a parallelogram $B = B(v; \tau M)$ for $M \in \mathbb{N}$, $\tau \in (1, \infty)$ and $v \in V$. Let s_T, s_B, s_L, s_R denote the top, bottom, left and right sides of B. Let $\mathcal{C} = \{c_i\}_{1 \leq i \leq j}$ be a set of j disjoint arms in A such that for each i, all of the vertices of c_i are open or all of them are closed. Let z_i be the endpoint of c_i on $\partial B(v; \tau M)$. Let $\eta \in (0, 1]$, we attach a parallelogram r_i to z_i as follows:*

$$r_i = \begin{cases} z_i + [-\eta M, \eta M] \boxtimes \left[0, 2\sqrt{\eta} M\right] & \text{if } z_i \in s_T \\ z_i + [-\eta M, \eta M] \boxtimes \left[0, -2\sqrt{\eta} M\right] & \text{if } z_i \in s_B \\ z_i + \left[-2\sqrt{\eta} M, 0\right] \boxtimes [-\eta M, \eta M] & \text{if } z_i \in s_L \\ z_i + \left[0, 2\sqrt{\eta} M\right] \boxtimes [-\eta M, \eta M] & \text{if } z_i \in s_R. \end{cases}$$

We say that $\mathcal{C}$ is η-well-separated on the outside, if the two following conditions are satisfied:

1. *The extremities* z_i $i = 1, 2, \ldots, j$ *are neither too close to each other:*

$$\forall i \neq l,\, d(z_i, z_l) \geq 10\sqrt{\eta}M,$$

nor too close to the corners Z_l $l = 1, 2, 3, 4$ *of* B *:*

$$\forall i, j,\, d(z_i, Z_l) \geq 10\sqrt{\eta}M.$$

2. *Each* r_i *is crossed vertically when* $z_i \in s_T \cup s_B$*, and horizontally when* $z_i \in s_L \cup s_R$ *by some crossing* $\tilde{c}_i$ *of the same colour as* c_i*, and*

$$c_i \text{ is connected to } \tilde{c}_i \text{ in } z_i + A\left(1, \sqrt{\eta}M\right).$$

We say that a set $\mathcal{C} = \{c_i\}_{1\leq i\leq j}$ of disjoint arms in A *can be made* η*-well-separated* on the outside, if there exists an set $\mathcal{C}' = \{c_i'\}_{1\leq i\leq j}$ of disjoint arms in A which is η-well-separated on the outside, and c_i' has the same colour and endpoint on $\partial B(v; M)$ as c_i for $i = 1, 2, \ldots, j$.

Similarly to Definition 5.A.1, we define the η-well-separation on the inside. The following statement follows from Lemma 15 of [77].

Lemma 5.A.2. *For* $\tau \in (1, \infty)$ *, and* $\delta > 0$*, there exists* $\eta(\delta) > 0$ *such that for any positive integer* N*, we have*

$$\mathbb{P}_{1/2}\left(\begin{array}{c}\textit{any set of disjoint arms in } A(N, \tau N)\\ \textit{can be made } \eta\textit{-well-separated on the outside}\end{array}\right) \geq 1 - \delta.$$

Moreover, the same statement holds for well separated arms on the inside.

We prove the following proposition.

Proposition 5.A.3. *Let* $k, N \in \mathbb{N}$*,* $a \in (10, \infty)$ *, and* σ *a colour sequence of length* k*. We divide the annulus* $A(N, aN)$ *into the annuli* $A_i = A\left(2^i N, 2^{i+1}N\right)$ *for* $i = 0, 1, \ldots, \lfloor \log_2(a) \rfloor - 1$*. Let* W *denote the set of indices* i *for which all the arms in* A_{3i+1} *wind around the origin at least once in the counter-clockwise direction for* $i = 0, 1, \ldots, \lfloor \log_2(a)/3 \rfloor - 1$*. There are positive constants* $c = c(k)$ *,* $\varepsilon = \varepsilon(k)$ *and* $N_0 = N_0(k)$ *such that*

$$\mathbb{P}_{1/2}\left(\mathcal{A}_{k,\sigma}(N, aN), |W| \geq c\log_2 a\right) \geq \left(1 - a^{-\varepsilon}\right)\pi_{k,\sigma}(N, aN)$$

for all $a \in (1, \infty)$ *and* $N \geq N_0$.

Remark 5.A.4. Proposition 5.2.7 follows from Proposition 5.A.3, since $W = \emptyset$ on the event $\mathcal{A}_{k,l,\sigma}(N, aN)$ when $l \geq 1$.

Proof of Proposition 5.A.3. For $a \leq 2$, the statement is trivial. Hence in the rest of the proof we suppose that $a > 2$. Classical RSW techniques [50] give that for all $k \in \mathbb{N}$ there is $\varepsilon_1 = \varepsilon_1(k) > 0$ such that

$$\pi_{k,\sigma}(N, aN) \geq a^{-\varepsilon_1} \tag{5.A.1}$$

uniformly in $a \geq 2$, $N \geq 1$ and $\sigma \in \{o, c\}^k$.

Let $\eta \in (0, 1/10)$. Let IS_i (OS_i) denote the event that any set of disjoint arms of A_i can be made η-well-separated on the inside (outside). Let WS denote the set of indices $i \in \left\{0, 1, \ldots, \left\lfloor \frac{\log_2 a}{3} \right\rfloor - 1\right\}$ for which OS_{3i} and IS_{3i+2} both hold. Notice that the events $\{i \in WS\}$ for $i = 1, 2, \ldots, \left\lfloor \frac{\log_2 a}{3} \right\rfloor - 1$ are independent. Moreover, by Lemma 5.A.2, for any $\delta > 0$ there is $\eta(\delta) \in (0, 1/10)$ such that

$$\mathbb{P}_{1/2}(i \in WS) \geq 1 - \delta.$$

Combining this with Hoeffding's inequality we get that c_0, δ, η such that

$$\mathbb{P}_{1/2}(|WS| \leq c_0 \log a) \leq a^{-2\varepsilon_1}. \tag{5.A.2}$$

This and (5.A.1) gives that

$$\begin{aligned}
\mathbb{P}_{1/2}(\mathcal{A}_{k,\sigma}(N, aN) \cap \{|WS| > c_0 \log(a)\}) \\
&\geq \pi_{k,\sigma}(N, aN) - \mathbb{P}_{1/2}(|WS| \leq c_0 \log a) \\
&\geq \pi_{k,\sigma}(N, aN) - a^{-2\varepsilon_1} \\
&\geq \left(1 - a^{-\varepsilon_1}\right) \pi_{k,\sigma}(N, aN) \qquad (5.A.3)
\end{aligned}$$

for all N.

Let us fix an integer $i \in \left\{0, 1, \ldots, \left\lfloor \frac{\log_2 a}{3} \right\rfloor - 1\right\}$. Condition on the event $\mathcal{A}_{k,\sigma}\left(N, 2^{3i+1}N\right) \cap \mathcal{A}_{k,\sigma}\left(2^{3i+2}N, aN\right) \cap \{i \in WS\}$ and on the configuration in $A(N, aN) \setminus A_{3i+1}$. This conditioning gives that all the arms in A_{3i} can be made η-well-separated on the outside, and all the arms in A_{3i+2} can be made η-well-separated on the inside. This imposes some conditions on the configuration in A_{3i+1} : there is a finite collection of disjoint parallelograms in which certain crossing events have to be satisfied. In order to have k arms with colour sequence σ in $A(N, aN)$, it is enough to connect, with the right colour, the k-tuple of parallelograms corresponding to the well separated versions of these arms on the inner parallelogram to those on the outer parallelogram of A_{3i+1}. There might be more than one choice for this pair of k-tuples of parallelograms. In this case we choose a pair in some deterministic way.

We connect the corresponding pairs of parallelograms by disjoint tubes of width $\sqrt{\eta}2^{3i+1}N$ in A_{3i+1} as in the proof of Lemma 4 of [68] (see Figure 9 of [68]), with the difference that these connections are special: We chose these tubes such that each of them winds around the origin at least twice in the counter-clockwise direction. We add an additional tube which avoids the ones above, connects the boundaries of the inner and the outer parallelograms of A_{3i+1} and winds around the origin at least twice in the counter-clockwise direction.

With standard RSW techniques one can show that the probability of the event that the original tubes are crossed in the hard direction by a path with the appropriate colour, and the additional tube is crossed in the hard direction with an open and a closed path is at least $h > 0$. Here $h = h(k, \eta)$ is independent of i, N and the location of the parallelograms we connected. The open and closed crossings of the additional tube forces all the arms of $A(N,aN)$ to wind around the origin in A_{3i+1} at least once in the counter-clockwise direction. Hence the event $\{i \in W\}$ occurs.

Thus the probability of $\{i \in W\}$ conditioned on the event $\mathcal{A}_{k,\sigma} \cap \{i \in WS\}$ and on the configuration in $A(N,aN) \setminus A_{3i+1}$ is at least h. Note that the event $\{i \in W\}$ only depends on the configuration in A_{3i+1}. Hence, when we condition on the event $\mathcal{A}_{k,\sigma}(N,aN)$ and on the realization of WS, the set W stochastically dominates a set Z, where the elements of Z are sampled from WS independently from each other with probability h.

Hence for $c > 0$ we have

$$\begin{aligned}
&\mathbb{P}_{1/2}\left(|W| \geq c\log_2 a \,|\mathcal{A}_{k,\sigma}(N,aN)\right) \\
&\quad \geq \mathbb{P}_{1/2}\left(|W| \geq c\log_2 a, |WS| \geq c_0 \log_2 a \,|\mathcal{A}_{k,\sigma}(N,aN)\right) \\
&\quad = \sum_S \mathbb{P}_{1/2}\left(|W| \geq c\log_2 a \,|\mathcal{A}_{k,\sigma}(N,aN), \, WS = S\right) \\
&\qquad\qquad\qquad\qquad \mathbb{P}_{1/2}\left(WS = S \,|\mathcal{A}_{k,\sigma}(N,aN)\right) \\
&\quad \geq \sum_S \mathbb{P}_{1/2}\left(|Z| \geq c\log_2 a \,|\mathcal{A}_{k,\sigma}(N,aN), \, WS = S\right) \\
&\qquad\qquad\qquad\qquad \mathbb{P}_{1/2}\left(WS = S \,|\mathcal{A}_{k,\sigma}(N,aN)\right),
\end{aligned} \tag{5.A.4}$$

where the summation over $S \subseteq \left\{0, 1, \ldots \left\lfloor \frac{\log_2 a}{3} \right\rfloor - 1\right\}$ with $|S| \geq c_0 \log_2 a$. We split this sum in (5.A.4) depending on the number of elements of S, and we get

$$\begin{aligned}
&\mathbb{P}_{1/2}\left(|W| \geq c\log_2 a \,|\mathcal{A}_{k,\sigma}(N,aN)\right) \\
&\quad \geq \mathbb{P}\left(Y \geq c\log_2 a\right) \sum_{l \geq c_0 \log_2 a} \mathbb{P}_{1/2}\left(|WS| = l \,|\mathcal{A}_{k,\sigma}(N,aN)\right) \\
&\quad = \mathbb{P}\left(Y \geq c\log_2 a\right) \mathbb{P}_{1/2}\left(|WS| \geq c_0 \log_2 a \,|\mathcal{A}_{k,\sigma}(N,aN)\right),
\end{aligned} \tag{5.A.5}$$

where Y is a random variable with distribution $Binom\left(c_0 \log_2 a, h\right)$. Using Hoeffding's inequality, we set $c = c(h), \varepsilon_2(h) > 0$ such that

$$\mathbb{P}\left(Y \geq c\log_2 a\right) \geq 1 - a^{-\varepsilon_2}. \tag{5.A.6}$$

By substituting (5.A.6) and (5.A.3) to (5.A.5) we get that

$$\mathbb{P}_{1/2}\left(|W| \geq c\log_2 a \,|\mathcal{A}_{k,\sigma}(N,aN)\right) \geq \left(1 - a^{-\varepsilon_1}\right)\left(1 - a^{-\varepsilon_2}\right)$$

for all $a > 2$ and N, which finishes the proof of Proposition 5.A.3. □

With suitable adjustments of arguments above, one can show that the following generalization of Proposition 5.2.7 holds.

Proposition 5.A.5. *For any $k \in \mathbb{N}$, there are positive constants $c = c(k)$, $\varepsilon = \varepsilon(k)$ such that for all $l, l' \in \mathbb{N}$ with $0 \leq l \leq l' \leq k$*

$$\pi_{k,l,\sigma}\left(n_0(k), N\right) \leq cN^{-\varepsilon} \pi_{k,l',\sigma}\left(n_0(k), N\right)$$

uniformly in N and in the colour sequence σ.

5.A.2 Existence of long thick paths in nice regions

Recall the Definition 5.4.3 and 5.4.4. First we prove Lemma 5.4.5 which is the special case of Lemma 5.4.6 where C is (a, b)-nice. Then we show how to modify the proof of Lemma 5.4.5 to deduce Lemma 5.4.6.

Lemma 5.4.5. *Let $a, b \in \mathbb{N}$ with $a \geq 2000$. Let C be an (a, b)-nice subgraph of $\mathbb{T}$. Then there is a $\lfloor a/200 - 10 \rfloor$-gridpath contained in C with diameter at least* $\operatorname{diam}(C) - 2b - 2a - 12$.

Remark 5.A.6. We believe that the constants in Lemma 5.4.5 are not optimal.

Proof of Lemma 5.4.5. Recall the lines below Definition 5.4.4. To prove Lemma 5.4.5, it is enough to find a path ζ in C such that $\operatorname{diam}(\zeta) \geq d - 2b - 2a - 12$ and $\zeta + B(a/100 - 5) \subset C$. We construct ζ by the following strategy.

We put hexagons on the vertices of $\mathbb{T}$ in the following way: The hexagon corresponding to the vertex v is the regular hexagon with side length $1/\sqrt{3}$ centred around v with one of its sides is vertical. These hexagons give a tiling of the plane $\mathbb{R}^2$. Using this tiling, we look at C as the region in $\mathbb{R}^2$ which is the union of the hexagons which are centred around the vertices of C.

Let $x, y \in C$ such that $d(x, y) = \operatorname{diam}(C)$. Let $\gamma \subset \mathbb{R}^2$ be a shortest curve connecting x and y in the region C, that is, γ is a continuous map of $[0, 1]$ such that 0 is mapped to x and 1 is mapped to y. We get the path ζ from γ as follows. First we cut down two pieces of γ one from its beginning and one from its end. We call the resulting path γ^2. Then we walk along γ^2, and if there is a point of ∂C 'close by' on the left (right) of γ^2, then we make a 'small' detour to the right (left). We get the path ζ from γ^2 after these detours. We show that ζ indeed satisfies the conditions above, and finish the proof of Lemma 5.4.5.

We gave a strategy which involved continuous curves and regions in the plane $\mathbb{R}^2$. We adapt it to the triangular lattice in the following precise proof.

Let $x = (x_1, x_2), y = (y_1, y_2) \in C$ such that $d(x, y) = \operatorname{diam}(C)$. We further assume that $x_1 < y_1$ and $d(x, y) = y_1 - x_1$. The other case where $d(x, y) = y_2 - x_2$ can be treated similarly. Let $\tilde{\gamma}$ denote a shortest (having the least number of vertices) path connecting x and y in C.

Note that there are $\binom{2n}{n}$ shortest paths between the vertices 0 and $n\underline{e}_1 + n\underline{e}_2$ in $\mathbb{T}$. However, most of them do not follow closely the straight line between the points 0 and $n\underline{e}_1 + n\underline{e}_2$. Hence $\tilde{\gamma}$ usually does not resemble a shortest continuous curve connecting x and y.

Step 1. *We choose a specific shortest path between x and y.*

For $u, v \in \mathbb{T}$, let $s(u, v)$ denote the line segment connecting u and v in $\mathbb{R}^2$. This segment naturally induces an oriented path $\sigma(u, v)$ in $\mathbb{T}$ as a sequence of the midpoints of the hexagons which are intersected by $s(u, v)$ as we walk along it from u to v. Note that it can happen that the segment $s(u, v)$ contains a side of a hexagon. In this case, we put only one of the neighbouring hexagons to $\sigma(u, v)$. We say that $\sigma(u, v)$ is a triangular grid approximation of the segment $s(u, v)$. Note that $\sigma(u, v)$ is a shortest path between u and v in $\mathbb{T}$.

Recall the notation in Section 5.4.1. Let $v, u, u' \in \tilde{\gamma}$ with $v \prec u, u'$ and $u \sim u'$. Then for all $w \in \sigma(v, u)$ there is $w' \in \sigma(v, u')$ with $w \sim w'$. Hence for $v \in \tilde{\gamma}$ there are two cases:

- either $\forall u \in \tilde{\gamma}_{v,y} \setminus \{v\}$ we have $\sigma(v, u) \setminus \{v\} \nsim \partial C$, or
- $\exists w = w(v) \in \tilde{\gamma}_{v,y} \setminus \{v\}$ such that $\forall u \in \tilde{\gamma}_{v,w} \setminus \{v, w\}$ we have $\sigma(v, u) \setminus \{v\} \nsim \partial C$, but $\sigma(v, w) \setminus \{v\} \sim \partial C$.

We perform the following procedure. We start at x. If the first case above holds for $v = x$, then we replace $\tilde{\gamma}$ by $\sigma(x, y)$ and finish the procedure. In the second case we replace $\tilde{\gamma}_{x,w(x)}$ by $\sigma(x, w(x))$, and repeat the procedure for $\tilde{\gamma}_{w(x),y}$ starting from $w(x)$. At each step of the procedure, we move at least one vertex further on $\tilde{\gamma}$, hence the procedure terminates in at most $|\tilde{\gamma}|$ steps. Let γ denote the path we get at the end. At each step of the procedure, we make modifications such that the new path is in C and its length is the same as the old path's. Hence $\gamma \subset C$ and $|\gamma| = |\tilde{\gamma}|$.

We finish Step 1 by with the following consequences of the construction above: γ resembles a shortest curve in $\mathbb{R}^2$: It is a sequence of triangular grid approximations of line segments in $\mathbb{R}^2$. Moreover, we have the following claim.

Claim 5.A.7. *As we walk along γ, we turn to the left (right) at $v \in \gamma$ if it has a neighbour in ∂C on the left (right) of γ. That is, if $u, v, w \in \gamma$ with $u \prec v \prec w$ and $\sigma(u, v), \sigma(v, w) \subset \gamma$, with $\sigma(u, v) \cup \sigma(v, w) \neq \sigma(u, v)$, then $v \sim \partial C \cap T(u, v, w)$, where $T(u, v, w)$ denotes the triangle spanned by the vertices u, v, w.*

Step 2. *We introduce some notation and assign labels to some of the vertices of γ.*

Let

$$ST := \{v = (v_1, v_2) \in V \mid x_1 < v_1 < y_1\}.$$

By possible shortening γ and redefining x and y, we can assume that $\gamma \subset cl(ST)$, $\gamma \cap \partial ST = \{x, y\}$ and $d(x, y) = \operatorname{diam}(C)$.

We set $\alpha := \lfloor a/6 \rfloor - 2 > 0$, and define

$$ST^i := \{v = (v_1, v_2) \in V \mid x_1 + b + i\alpha < v_1 < y_1 - b - i\alpha\}$$

for $i \in \{1, 2\}$. Let x^i (y^i) denote the last (first) vertex of γ which is in the half plane $\{v = (v_1, v_2) \in V \mid v_1 \leq x_1 + b + i\alpha\}$ ($\{v \mid v_1 \geq x_1 - b - i\alpha\}$). Let $\gamma^i = \gamma_{x^i, y^i}$. Note that $ST^1 \supset ST^2$ and γ^2 is a subpath of γ^1.

Let $i \in \{1, 2\}$. Since γ^i is a shortest path, it is non self-touching. This combined with $\gamma^i \cap \partial ST^i = \{x^i, y^i\}$ we get that γ^i, cuts $cl(ST^i)$ into two connected components. Let ${ST^i}_L$ (${ST^i}_R$) denote connected component $cl(ST^i) \setminus \gamma^i$ which is on the left (right) had side of γ^i as we walk along it.

For $v \in \gamma^2$, we put a label $l(v) \in \{L, R, N, G\}$ as follows. We denote the set of vertices with label $X \in \{L, R, N, G\}$ by γ^2_X. First we define the labels R and L : For $v \in \gamma^2$, we set $l(v) = L$ ($l(v) = R$) if ${ST^1}_L \cap B(v; \alpha) \cap \partial C \neq \emptyset$

$(ST^1{}_R \cap B(v;\alpha) \cap \partial C \neq \emptyset)$. To show that the labels L, R are well-defined, we have to check that for $v \in \gamma^2$ at most one of the sets $ST^1{}_L \cap B(v;\alpha) \cap \partial C$ and $ST^1{}_R \cap B(v;\alpha) \cap \partial C$ is non-empty. Since $2\alpha < a$, this follows from Condition 3 of Definition 5.4.4. Let $\beta := \lfloor \alpha/3 \rfloor$. For $v \in \gamma^2 \setminus \left(\gamma_L^2 \cup \gamma_R^2\right)$ we set $l(v) = G$ if $B(v;\beta) \cap \left(\gamma_L^2 \cup_R^2\right) = \emptyset$, and $l(v) = N$ otherwise.

Since $4\alpha + 2\beta < a$, it is a simple exercise to prove the following claim using Condition 3 of Definition 5.4.4, which finishes Step 2.

Claim 5.A.8. *Let $u \in \gamma_L^1$ and $v \in \gamma_R^1$. Then there is $w \in \gamma_G^1$ which is in between u and v.*

Step 3. *We define the neighbourhoods $\mathcal{F}_v$ and $\mathcal{G}_v$ for $v \in \gamma^2$.*

If $l(v) \in \{G, N\}$ then we set $\mathcal{F}_v := B(v;\alpha)$ and $\mathcal{G}_v := B(v;\beta)$.

If $l(v) \in \{L, R\}$, let f^1 (f^2) as the last vertex when we go backwards (forward) from v along γ which is in $B(v;\alpha)$. If it has label L (R) then we define $\mathcal{F}_v$ as the connected component of $B(v;\alpha) \setminus \gamma_{f^1,f^2}$ on the right (left) hand side of γ_{f^1,f^2}. Similarly we define g^1 and g^2 in the box $B(v;\beta)$, and $\mathcal{G}_v$.

The combination of $4\alpha < a$, Claim 5.A.7 and Condition 3 of Definition 5.4.4 gives that

$$\left(\gamma_{f^1,g^1} \cup \gamma_{g^2,f^2}\right) \cap B(v;\beta - 1) = \emptyset.$$

Hence we get

Claim 5.A.9. *$\mathcal{F}_v \cap B(v;\beta) = \mathcal{G}_v$ for $v \in \gamma^2$.*

Step 4. *We investigate the neighbourhood $\mathcal{G}_v$.*

Claim 5.A.10. *$\mathcal{G}_v \cap \partial C = \emptyset$ for $v \in \gamma^2$, and $\mathcal{G}_v \cap \gamma^1 = \emptyset$ for $v \in \gamma_L^2 \cup \gamma_R^2$.*

Proof of Claim 5.A.10. First we show that $\mathcal{G}_v \cap \partial C = \emptyset$ with a proof by contradiction. Suppose that $\mathcal{G}_v \cap \partial C \neq \emptyset$. The definition of labels give that if $\mathcal{G}_v \cap \partial C \neq \emptyset$, then $l(v) = L$ or R. We further suppose that $l(v) = L$. The case where $l(v) = R$ can be treated similarly. We choose w so that it is one of the closest vertices to v among the vertices of $\mathcal{G}_v \cap \partial C$. See Figure 5.8.

By the definition of the label L, we have that $w \in ST_L^2 \cap B(v;\beta)$. Since $w \in \mathcal{G}_v$, i.e. w is on the right hand side of γ_{f^1,f^2} in $B(v;\alpha)$. Hence some subpath of $\gamma^1 \setminus \gamma_{f^1,f^2}$, denoted by ν, has to separate w from v in $\mathcal{F}_v$. Let us walk from v to w on $\sigma(v,w)$, till we hit ν. Let us denote the explored path by $\sigma(v,v')$, where v' is the last point of the exploration. Let γ' be the path we get when we replace the part of γ between v and v' by $\sigma(v,v')$. Consider the case $v' \prec_\gamma v$. The other case where $v' \succ_\gamma v$ can be treated similarly. The number of vertices of $\sigma(v,v')$ is at most 2β. However, the number of vertices in ν before v' is at least $\alpha - \beta$. Moreover, $\left|\gamma_{f^1,v}\right| \geq \alpha - \beta$. Hence

$$\begin{aligned} |\gamma| - |\gamma'| &\geq 2(\alpha - \beta) - 2\beta \\ &\geq \frac{2}{3}\alpha > 0. \end{aligned} \tag{5.A.7}$$

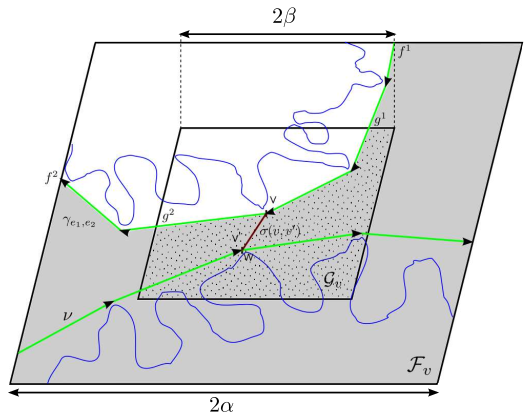

Figure 5.8: The path $\gamma_{x,v} \vee \sigma_{v,w} \vee \gamma_{w,y}$, is shorter than γ by at least $\frac{2}{3}\alpha$ vertices.

The definition of w gives that $\sigma(v, v') \subset C$, thus $\gamma' \subset C$. Hence γ' connects x and y in C and by 5.A.7, it is shorter than γ. This contradicts the definition of γ, hence $\mathcal{G}_v \cap \partial C = \emptyset$ for $v \in \gamma^2$.

The proof of $\mathcal{G}_v \cap \gamma^1 = \emptyset$ for $v \in \gamma^2$ is quite similar to the one above, hence we omit it, and finish the proof of Claim 5.A.10 and concludes Step 4. $\square$

Step 5. *We define the path* ζ.
We set $\varepsilon = \lfloor \beta/4 \rfloor - 2$. For $j \in \{L, R\}$, let

$$U_j := \bigcup_{v \in \gamma_j^2} B(v; \varepsilon). \tag{5.A.8}$$

$ST_R^2 \setminus U_L \setminus \gamma^2$ ($ST_L^2 \setminus U_R \setminus \gamma^2$) has one infinite connected component which we denote by Z_R (Z_L). Let ζ_j denote the shortest path in $\partial Z_j \cap ST^2$ which connects the left and the right side of ST_2. We orient ζ_L (ζ_R) so that Z_L (Z_R) is on the left (right) hand side. Note that ζ_L, ζ_R are left-right crossings of ST^2.

Note that ζ_L, ζ_R and γ^2 are non self-touching paths. Since Z_R, Z_L and γ^2 are disjoint, γ^2 is sandwiched between ζ_L and ζ_R. Hence $\zeta_L, \zeta_R, \gamma^2$ can have common vertices, but they cannot cross each other. Thus we get the following claim.

Claim 5.A.11. *Let* $v \in \zeta_L \cap \zeta_R$. *Then* $v \in \gamma^2$.

Condition 3 of Definition 5.4.4 implies the following claim.

Claim 5.A.12. *Let* $v \in \zeta_L \cap \zeta_R$. *If* w, *the next vertex after* v *on* γ^2 *exists, then* $w \in \zeta_L \cup \zeta_R$.

Let $\overrightarrow{G}=\left(\overrightarrow{V},\overrightarrow{E}\right)$ be the directed graph induced by the directed paths ζ_L,ζ_R and γ^2.That is $\overrightarrow{G}=\left(\overrightarrow{V},\overrightarrow{E}\right)$ where $\overrightarrow{V}=\zeta_L\cup\zeta_R\cup\gamma^2$, and $(u,v)\in\overrightarrow{E}$ if and only if $u,v\in\nu$, $u\sim v$ and $u\prec_\nu v$ for some $\nu\in\left\{\zeta_L,\zeta_R,\gamma^2\right\}$. Using the definition of ζ_L and ζ_R it is a simple exercise to show the following claim.

Claim 5.A.13. *$\overrightarrow{G}$ has no directed loops.*

For $j\in\{L,R\}$ and $z\in\zeta_j$ let $n_j(z)$ be the first vertex of $\zeta_j\cap\gamma^2$ after z on ζ_j. That is, $n_j(z)\in\zeta_j\cap\gamma^2$ with $n_j(z)\succeq_{\zeta_j} z$ and if $z'\in\zeta_j\cap\gamma^2$ with $z'\succ_{\zeta_j} z$ then $z'\succeq_{\zeta_j} n_j(z)$. If there is no such vertex, then we set $n_j(z)=\emptyset$.

We define a directed path ζ by the following procedure. Let z_j denote the starting point of ζ_j for $j\in\{L,R\}$. ζ starts at the vertex z defined as

$$z:=\begin{cases} z_L & \text{when } n_L(z_L)=\emptyset,\\ z_L & \text{when } n_L(z_L)\neq\emptyset\neq n_R(z_R) \text{ and } n_L(z_L)\succeq_{\gamma^2} n_R(z_R)\\ z_R & \text{otherwise.}\end{cases}$$

Suppose that we are at vertex v in ζ. If v is the endpoint of ζ_L or ζ_R, we terminate the procedure. Otherwise, we define the next vertex of ζ, denoted by w, as follows. For $j\in\{L,R\}$, if $v\in\zeta_j$, then v_j denotes the next vertex after v in ζ_j.

- If $v\in\zeta_L\setminus\zeta_R$, then $w=v_L$
- if $v\in\zeta_R\setminus\zeta_L$, then $w=v_R$
- if $v\in\zeta_L\cap\zeta_R$, and if
  - $v_L,v_R\in\gamma^2$, then the definition of ζ_L and ζ_R gives that $v_L=v_R$ and we take $w=v_L=v_R$
  - $v_L\in\gamma^2$, $v_R\notin\gamma^2$, then $w=v_R$
  - $v_R\in\gamma^2$, $v_L\notin\gamma^2$, then $w=v_L$
  - the case $v_L,v_R\notin\gamma^2$ is impossible by Claim 5.A.12.

We finish Step 5 by showing that ζ is well-defined. The definition of ζ shows that if we view ζ as a directed graph, it is a subgraph of $\overrightarrow{G}$. Hence by Claim 5.A.13 ζ has no directed loops. Thus ζ is self avoiding, and the procedure above terminates after finitely many steps, when ζ reaches the endpoint of ζ_L or ζ_R.

Step 6. *We prove the following claim and finish the proof of Lemma 5.4.5.*

Claim 5.A.14. *$\zeta+B(\varepsilon)\subset C$ and* $\operatorname{diam}(\zeta)\geq d(x,y)-2b-4\alpha$.

Proof of Claim 5.A.14. The definition of ζ shows that ζ is a horizontal crossing of ST^2. Hence $\operatorname{diam}(\zeta) \geq d(x,y) - 2b - 4a$. We show that for all $v \in \zeta$ we have $v + B(\varepsilon) \subset C$. There are two cases depending on whether v is contained in γ^2.

Case 1: $v \in \zeta \setminus \gamma^2$. Then $v \in \zeta_L \setminus \gamma^2$ or $v \in \zeta_R \setminus \gamma^2$. We assume that $v \in \zeta_L \setminus \gamma^2$. The case where $v \in \zeta_R \setminus \gamma^2$ can be treated similarly. The definition of ζ_L gives that there is $w \in \gamma_R$ such that $v \in (B(w;\varepsilon+1) \setminus B(w;\varepsilon))$ and $B(v;\varepsilon) \cap \gamma_R = \emptyset$. This combined with $4\alpha + 4\varepsilon + 2 < a$ and Condition 3 of Definition 5.4.4 gives that $B(v;\varepsilon) \cap \left(\gamma_L^2 \cup \gamma_R^2\right) = \emptyset$.

If $\gamma^2 \cap B(v;\varepsilon) \neq \emptyset$, then $\exists u \in \left(\gamma_G^2 \cup \gamma_N^2\right) \cap B(v;\varepsilon)$. Claim 5.A.10 implies that $C \supset \mathcal{G}_u = B(u;\beta) \supset B(v;\varepsilon)$ since $4\varepsilon < \beta$.

If $\gamma^2 \cap B(v;\varepsilon) = \emptyset$, then the definition of w and Claim 5.A.10 shows that $C \supset \mathcal{G}_w \supset B(v;\varepsilon)$ since $2\beta + 2\varepsilon < \alpha$.

Hence $B(v;\varepsilon) \subset C$ in Case 1.

Case 2: $v \in \zeta \cap \gamma^2$. Since $\zeta \subset \zeta_L \cup \zeta_R$, we assume that $v \in \zeta_L$. The case where $v \in \zeta_R$ can be treated similarly. First we show that $v \notin \gamma_L^2 \cap \zeta_L$.

Suppose the contrary, that is $v \in \gamma_L^2 \cap \zeta_L$. Let w be the starting point of the connected component of v in $\gamma^2 \cap \zeta$. By the definition of ζ, $w \in \zeta_L$. Moreover, for w' the vertex right before w on ζ_L, we have $w' \in \zeta_L \setminus \gamma_2$. Hence there is $u' \in \gamma_R^2$ such that $w' \in B(u';\varepsilon+1)$. Since $v \in \gamma_L$ and $u' \in \gamma_R^2$, by Claim 5.A.8 $\exists u \in \gamma_G^2$ which is between u' and v on γ^2. Note that $w' \in \mathcal{G}_{u'}$. By Claim 5.A.10 we have that $\gamma^2_{u',w} \subset \gamma^2 \setminus \gamma_G^2$. Hence u is between w and v on γ^2. From the definition of w, we get that $u \in \zeta \cap \zeta_L$.

Note that if we show that $u \in \zeta_R$, then we get a contradiction by the definition of ζ. Hence in order to rule out the case $v \in \gamma_L^2 \cap \zeta_L$ it is enough to show that $u \in \zeta_R$.

Suppose the contrary, that is $u \notin \zeta_R$. Recall the definition of U_L from 5.A.8. We introduce a new set of labels on the vertices of U_L as follows. For $q \in U_L$ there is a vertex $r \in \gamma_L$ such that $q \in B(r;\varepsilon)$. We define

$$l'(q) := \begin{cases} B & \text{if } r \prec_{\gamma^2} u \\ A & \text{otherwise.} \end{cases}$$

Since the choice of r above is not necessarily unique, we have to show that $l'(q)$ is well-defined. It can be easily checked by combining Claim 5.A.10, $4\varepsilon + 4 < \beta$ and $u \in \gamma_G^2$. Moreover a similar argument shows that if $q, q' \in U_L$ with $q \sim q'$, then $l'(q) = l'(q')$.

Since γ^2 is non self-touching, $u \in \gamma^2$ is connected to ∞ in ST_R. Since $u \notin \zeta_R$ it is not connected to ∞ in Z_R, there is a path $\nu \subset U_L$ which separates u from ∞ in ST_R. We can choose ν such that it starts and ends at a vertex neighbouring γ^2. By a possible shortening of ν, we can assume that if $u' \in \nu$ with $u' \sim \gamma^2$, than u' is either the starting or the endpoint of ν. Let u_1, u_2 be neighbours of the starting point and the endpoint of ν which are in γ^2. The definition of ν gives that u is in between u_1 and u_2 on γ^2. Using Condition 3 of Definition 5.4.4 and that $u \in \gamma_G^2$ it is easy to check that $l'(u_1) \neq l'(u_2)$.

On the other hand, ν is a connected subset of U_L, hence l' is constant on ν. This is a contradiction, thus $u \in \zeta_R$, which in turn shows that $v \in \gamma_L^2 \cap \zeta_L$.

Hence $v \notin \gamma_L^2 \cap \zeta_L$ but $v \in \zeta \cap \gamma^2 \cap \zeta_L$. The definition of ζ_L gives that $v \notin \gamma_R$. Hence $v \in \gamma_N^2 \cup \gamma_G^2$. By Claim 5.A.10 we get $C \supset \mathcal{G}_v = B(v;\beta) \supset B(v;\varepsilon)$, and we are done in Case 2. Since there are no other cases left, the proof of Claim 5.A.14 is finished. $\square$

Since $\zeta + B(\varepsilon) \subset C$ and $\operatorname{diam}(\zeta) \geq d(x,y) - 2b - 4\alpha$ hence the $\lfloor \varepsilon/2 \rfloor$-gridpath approximation of ζ is contained in C. It has diameter at least $d(x,y) - 2b - 4\alpha - \varepsilon \geq d(x,y) - 2b - 2a - 12$. Since $\varepsilon = \lfloor \beta/4 \rfloor - 2 \geq a/100 - 5$ this concludes the proof of the Lemma 5.4.5. $\square$

We finish the appendix by proving Lemma 5.4.6.

Lemma 5.4.6. *Let $a, b, c \in \mathbb{N}$ with $a \geq 2000$. Let C be subgraph of $\mathbb{T}$ which is (a,b)-nice in $B(c)$. Let C' be a connected component of $C \cap B(c)$. Then there is a $\lfloor a/200 - 10 \rfloor$-gridpath contained in C' with diameter at least* $\operatorname{diam}(C') - 2b - 2a - 12$.

Proof of Lemma 5.4.6. Let $x, y \in C'$ with $d(x,y) = \operatorname{diam}(C')$. We choose $\tilde{\gamma}$ as one of the shortest paths connecting x, y in C'. From this point on, we can follow the proof of Lemma 5.4.5 since we will use Condition 3 of Definition 5.4.4 for pairs of vertices $u, v \in \partial C$ which are contained in $B(c)$. $\square$

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Acknowledgements

I was fortunate to have many colleagues and fellow researchers who helped me in my research which lead to this thesis. It is an impossible task to thank each and every one of them individually. Even if I did my best, the acknowledgement would take more than a couple of pages, and there still would be plenty of excellent colleagues left out. Nevertheless, I thank all of you for the support and the advice you gave me throughout the years of my PhD.

I am grateful to my supervisor Jacob van den Berg for his guidance. I will always remember him for his attention to detail, for the clarity of his mind and for his excellent taste in problems. I thank him for introducing me to the scientific community and for his patience especially in those times when he read the early versions of my drafts and papers. It was a pleasure to work with him and learn from him.

I am thankful to the members of my reading committee, Federico Camia, Roberto Fernandez, Frank den Hollander and Pierre Nolin for devoting their time to read my thesis and for their valuable comments. I am grateful to my co-authors Jacob van den Berg, Federico Camia, René Conijn and Pierre Nolin. Without your ideas and insight I would have never finished my PhD. I hope our cooperation will continue in the future. Special thanks goes to Franz Merkl, Artëm Sapozhnikov and Vladas Sidoravicius for inviting me for research visits. It was a great experience to discuss mathematics with you. I look froward to see you again. I am grateful to Balint Tóth for arranging a desk for me for a month in Budapest. It was a excellent opportunity to keep in touch with old friends and meet fellow mathematicians. I thank Marie-Colette van Lieshout and Artëm Sapozhnikov for their advice on research in general. I am grateful to Federico Camia for sharing his insight into statistical mechanics, and for proving to me that one of the most efficient ways to do mathematics is by having dinner with the right people.

My dear colleagues at CWI and at VU, it was a pleasure to spend these years together. Thank you for the chats, dinners, borrels, barbecues and game nights we shared. You made my stay in the Netherlands enjoyable. Matthijs, I am grateful to you for translating my summary to Dutch on such a short notice. Joost and Martijn, thank you for helping me with my computer and programming problems, you saved many hours of my time. Masha and Kuba, thank you for being my paranymphs. I thank the support and administrative staff of CWI and VU for their continued help, even though their work is only noticed when there is a problem.

Last, but not least, I thank my family and friends for their constant support and their encouragement in times when it was needed the most.

Samenvatting

Onderwerpen in percolatietheorie

Dit proefschrift behandelt verschillende onderwerpen die te maken hebben met het percolatiemodel. Dit is een model voor een poreuze steen, waarbij de gaten van de steen voorgesteld worden als kanalen die op regelmatige wijze met elkaar verbonden zijn. De kanalen laten water op de volgende manier door. Zij $p \in [0,1]$ de parameter van het model. Elk kanaal is open, dat wil zeggen dat het water doorlaat, met kans p, onafhankelijk van andere kanalen, en gesloten met kans $1-p$. Net zoals bij andere modellen uit de statistische mechanica zijn we voornamelijk geïnteresseerd in het gedrag op grote schaal, de zogeheten macroscopische eigenschappen van het model. Dit houdt in dat we ons concentreren op het geval waarin de kanalen veel kleiner zijn dan de steen zelf. Het blijkt dat er een kritieke parameter $p_c \in [0,1]$ bestaat, zó dat als $p < p_c$, er louter kleine gaten zijn. Echter, als $p > p_c$, dan lijkt de steen op een spons: wanneer we de steen onderdompelen in water, zal het water het midden van de steen bereiken.

Het verschijnsel dat we hierboven beschreven hebben wordt een zogenaamde faseovergang genoemd. Een ander voorbeeld van dit verschijnsel is het gedrag van water bij verschillende temperaturen: onder 0 graden Celsius is het ijs, daarboven is het vloeibaar water. Beschouw het percolatiemodel nu in een breder verband.

We zien dat verschillende kleine schaal eigenschappen (de kans dat een kanaal open is) kunnen leiden tot verschillend gedrag op grote schaal (of het water het midden van de steen bereikt of niet). Oftewel, het percolatiemodel ondersteunt het idee dat atoom-schaal eigenschappen de globale eigenschappe van materie kunnen bepalen (of de materie vast, vloeibaar of gas is). Een ander voorbeeld dat ondersteunt dit idee is het realistischer en nauw verwante Ising model voor magnetisme.

De eigenschappen van het percolatiemodel dat we hierboven beschreven hebben, rechtvaardigen verder onderzoek. In het bijzonder is het gedrag van het model op het kritieke punt $p = p_c$ interessant. In de afgelopen twintig jaar is er substantiële vooruitgang geboekt in de analyse van het kritieke model in twee dimensies. Door resultaten uit complexe en stochastische analyse, meetkunde en grafentheorie te combineren, konden enkele opmerkelijke symmetrieën van het tweedimensionale kritieke model worden afgeleid. Desalniettemin staan er nog vele problemen open die eenvoudig te formuleren zijn.

Het doel van dit proefschrift is om bij te dragen aan de percolatietheorie. We verfijnen enkele technieken die gebruikt worden bij de bestudering van het model en onderzoeken enkele gerelateerde modellen. We beginnen met de beschrijving van enkele van de gebruikte technieken.

Herinner de beschrijving van de faseovergang van het percolatiemodel hierboven. Een wat gedetailleerdere analyse laat zien dat als de kanalen veel kleiner zijn dan de steen er sprake is van een zogeheten $0-1$-wet. Voor een kleine $\varepsilon > 0$ hebben we dat als $p > p_c + \varepsilon$, het water het midden van de steen bereikt met kans dichtbij 1, terwijl voor $p < p_c - \varepsilon$, dezelfde kans dichtbij 0 is. Het blijkt dat deze benaderende $0-1$-wet volgt uit enkele concentratie-ongelijkheden en scherpe-drempelwaarde-resultaten. Deze resultaten beschrijven enkele fundamentele eigenschappen van functies van vele onafhankelijke variabelen. Ze worden daarom in vele vakgebieden gebruikt, zoals in de informatica voor de analyse van probabilistische algoritmes of in de economie voor de analyse van kiessystemen. We motiveren onze resultaten door een voorbeeld te geven van zo'n kiessysteem. We nemen aan dat er n kiezers zijn en elke kiezer stemt 'ja' of 'nee' met kans $1/2$, onafhankelijk van andere kiezers. Er is een beslissingsproces dat beslist wat de uitkomst van de stemming is, gegeven de stemmen van de kiezers. De invloed van een individu i is de kans dat de uitkomst van de stemming verandert als het individu i zijn beslissing verandert, terwijl de andere stemmen ongewijzigd blijven. Een vermaard resultaat is dat, kort gezegd, de som van de invloeden het grootst is als de invloed van ieder individu klein is. Onze bijdrage hieraan is een generalisatie van de ongelijkheid van Talagrand, welke een gekwantificeerde versie is van het resultaat hierboven. Ons resultaat is strict genomen niet nieuw, maar het bewijs hiervan is anders dan de bestaande bewijzen in de literatuur.

Vervolgens bestuderen we het eerste-passage-percolatiemodel. Dit model kunnen we beschouwen als een uitbreiding van het percolatiemodel: we kennen nu een positief getal toe aan een kanaal, in plaats van het open of gesloten te noemen. Dit getal geeft de tijd aan die het water nodig heeft om door het kanaal te gaan. We voorzien de steen nu van water op een gegeven positie en onderzoeken hoe het water zich verspreidt in de steen. We geven een bovengrens voor de variantie van de snelheid waarmee het water zich verspreidt. Dergelijke bovengrenzen zijn al bekend in de literatuur voor het geval dat de passagetijden van verschillende kanalen onafhankelijk zijn van elkaar. Onze methode is vernieuwend omdat het enkele resultaten uit de literatuur uitbreidt naar het geval waarin de passagetijden zwak afhankelijk zijn. Onze gegeneraliseerde versie van de ongelijkheid van Talagrand speelt een cruciale rol in het bewijs van de genoemde bovengrens voor de variantie.

In het hierboven genoemde eerste-passage-percolatiemodel groeit het gebied waarover het water zich heeft verspreid naarmate de tijd toeneemt. We beschouwen een ander groeimodel als volgt. We beginnen met het percolatiemodel, waar de regelmatige structuur van de kanalen een oneindig systeem is, bijvoorbeeld de binaire boom of een rooster. Ons uitgangspunt is de versie van het percolatiemodel waar op tijdstip 0 alle kanalen zijn gesloten. Vervolgens elke kanaal, onafhankelijk van elkaar, opent op een willekeurige tijd die uniform verdeeld over het interval $[0, 1]$ is. We krijgen het volgende: op tijdstip $p = 0$ is ieder kanaal gesloten. Als we vervolgens p laten toenemen, openen steeds meer kanalen zich en beginnen grotere en grotere gaten (open clusters) te vormen. Voor $p > p_c$ zal een oneindig open cluster gevormd worden en op tijdstip $p = 1$ zijn alle kanalen geopend. We passen dit proces aan door open clusters waarvan de grootte groter dan N is, niet toe te staan door te groeien, waarbij N de parameter van het model is. Oftewel, open clusters met een grootte groter dan N 'bevriezen'. We onderzoeken het gedrag voor grote N van dit zogenaamde N-parameter-bevroren-percolatiemodel voor de gevallen waar de regelmatige structuur van de kanalen de binaire boom is en het tweedimensionale vierkantsrooster. Tenslotte merken we nog op dat er een verborgen parameter in het model is: de manier waarop we de clusters meten.

Beschouw vervolgens het proces op de binaire boom. We laten zien dat als N richting ∞ gaat, het N-parameter-bevroren-percolatiemodel, onder enkele milde voorwaarden voor de manier waarop we de clusters meten, convergeert naar het proces waarbij we, in de beschrijving van het model hierboven, N vervangen door ∞. De dynamica van dit ∞-parameter proces stuurt het model naar een toestand waarin het model op alle tijdstippen groter dan $1/2$ lijkt op het kritieke percolatiemodel op de binaire boom. Dit is een voorbeeld van het zogeheten zelforganiserende kritieke verschijnsel, wat het tot een vrij interessant model maakt: aardbevingen en fluctuaties in financiële markten beschouwt men ook als voorbeelden van dit verschijnsel.

In het geval van het vierkantsrooster is de situatie nogal anders, aangezien het ∞-parameter proces hierop niet bestaat. Bovendien hangt het gedrag van het N-parameter-bevroren-percolatiemodel af van de manier waarop we de clusters meten. We beperken ons daarom tot het geval waarin we de clusters meten met hun diameter. Het blijkt dat als N naar ∞ gaat, dat bevroren clusters slechts ontstaan op tijdstippen dichtbij $p_c = 1/2$ en op kritieke percolatieclusters lijken. Voorts geven onze resultaten precieze grenzen voor de tijden waarop bevroren clusters gevormd worden. Dit leidt ons tot het volgende vermoeden: kort gezegd, we verkrijgen een limietproces als we het N-parameter-bevroren-percolatiemodel dichtbij tijdstip $1/2$ bekijken, de ruimte met N schalen en de tijd met de hierboven verkregen grenzen schalen. We denken ook dat dit proces het gedrag voor grote N van de N-parameter processen volledig beschrijft. Dit vermoeden bewijzen en de N-parameter processen onderzoeken voor de gevallen waarin we de clusters op een andere manier meten, zijn uitdagingen voor toekomstig onderzoek.